ELEMENTARY GEOMETRY.

THE ELEMENTARY GEOMETRY OF THE RIGHT LINE AND CIRCLE,

FOR THE USE OF

SCHOOLS AND COLLEGES,

WITH EXERCISES.

BY

WILLIAM A. WILLOCK, D.D.,

FORMERLY FELLOW OF TRINITY COLLEGE, DUBLIN.

LONDON:

LONGMANS, GREEN AND CO.

1875.

LONDON:
PRINTED BY W. WILFRED HEAD, PLOUGH COURT, FETTER LANE, E.C.

PREFACE.

THIS treatise is published with a view to facilitate the learning of Geometry in its first stages. It is very generally acknowledged that Euclid's "Elements" is unfit, as a text-book, for the quick and effective education in Geometry of the youth of the present age. A treatise which would combine all the excellences of Euclid with the advantages of the nomenclature and methods of Modern Geometry, and place the subject before the student in a plain, natural order of development, has been very much desired. This work professes to be, at least, a fair approximation to the ideal of such a handbook. It, no doubt, has its faults; but, taken as a whole, it may, nevertheless, be a move in the right direction.

The course I have followed is, to take the materials of Euclid, to add to them, and, retaining his plan of stating every distinct theorem or problem in a formal proposition, to arrange the whole in what I should conceive to be a simple and natural order of demonstration—such an order as was suggested, two hundred years ago, by the distinguished Antoine Arnauld, in his "Port-Royal Logic"—and to do this, not in an antiquated and semi-syllogistic style, but in the

ordinary language in which all modern English mathematical books are written.

The first step was to separate the problems from the theorems, on the ground that the former are by no means necessary to the demonstration of the latter. The reasons for taking this course are given in the Second Note to the Appendix, the chief of which is, that problems not only derange the natural order of the theorems, but unnecessarily do so. Consequently, the problems in this treatise are given at the ends of the several chapters, excepting the first, which hardly furnishes material for a problem, being confined to the right line, or directive.

The course was thus open for a natural arrangement of the subject. After much consideration, and having in mind the acknowledged fact—"The reproach of geometry"—that Euclid never proved the twenty-ninth proposition of his First Book on any basis of postulation or assumption more evident than the proposition itself, I could not resist the conclusion that there was, in his system, some definition left out or important geometric idea overlooked. That idea was Direction. (*See Appendix, Note* 1.) Afterwards, I ascertained that "direction" had often been suggested by geometricians — as may be seen in the article on Parallels in the Penny Cyclopædia—as the true basis of a doctrine of parallel lines. Confirmed thus in my convictions, it became clearly evident to me that right lines, or directives, must contain within themselves, in their intersections, all their angular properties without any need of finite line, triangle, or circle, to make them manifest. Hence, the

First Chapter of this treatise is confined to the consideration of the directive, the simplest of all figures—if it may be called a "figure"—the distinctive quality of which is *sameness* of direction throughout.

The next, and only other, figure of which Elementary Geometry treats is the Circle, the natural function of which seems to be the furnishing of a comparative measure of the lengths of finite right lines, that is, of portions of directives—which be equal, which greater, and which less? Thus the directive should precede the circle; and the circle, which by itself has no property but symmetry of form, in its intersections with directives, and aided by the directive, should make manifest its own peculiar properties. This seems to me to be the natural and truly philosophical basis of an elementary geometry of the right line and circle; and, for that reason, the Second Chapter is given to the circle, which is first considered in reference to its intersections with directives and with other circles, and afterwards as to its tangencies.

The properties of directives and circles being thus established, the next step is to apply them to the determining of the relations of the sides and angles of a triangle to each other, and to the several cases of equal triangles—also to the relations of the sides and angles of parallelograms, which are but cases of two equal triangles on opposite sides of a diagonal. The isosceles triangle, the properties of which come immediately from the circle, takes the lead in these demonstrations, which are given in the Third Chapter.

In these three chapters there is no mention of *an area*. They are confined to the right line and circle

in their relations of angular and linear magnitudes, including those of the triangle and parallelogram. Area is, for the first time, introduced in the Fourth Chapter, in which it is considered, by the aid of Euclid's principle of superposition, in reference to areas *generally*, to the equality of the areas in the several cases of equal triangles, and of equal segments and sectors of circle—and, also, the equality of the areas of parallelograms and triangles on the same or equal bases and between the same parallels. Thus, in the first four chapters, almost the whole substance, theorems and problems, of Euclid's first, third, and fourth books are given.

Euclid's Second Book, placed between two comparatively easy ones, is out of place, and has been a sore stumbling-block to young beginners. Its subject-matter is here held back for the Fifth Chapter, not on account of this difficulty, but because the theorems and problems are more advanced, and should, therefore, in a natural order of demonstration, take later rank. This will be an advantage to the student, who will, moreover, find the subject more fully explained than it is in Euclid, and in two different points of view. Also, several propositions — the forty-seventh and forty-eighth of Euclid's first book, the thirty-fourth to the thirty-seventh, inclusive, of his third, and the tenth, eleventh, and sixteenth of his fourth—are transferred to the Fifth Chapter as their proper place, they being propositions relating to or depending on squares and rectangles. The forty-seventh, in this treatise, immediately precedes two theorems (Euclid's twelfth and thirteenth, Second Book) from which it should

never have been disassociated. It is evident that Euclid placed it in his First Book solely with a view to prove the ninth and tenth theorems of his Second, which, however, could have been easily proved without it. No other motive, at least, can be assigned for his taking it out of its natural place.

The Chapter on Proportion stands in its traditional place—the natural place—at the end of the treatise. The criterion of proportion used is that of Elrington, by *sub*multiples. This test is here adopted because it is more easily understood by young students, and also more conformable to the common notions of proportion. Moreover, it holds good, in all strictness, for commensurable magnitudes; and, as to the incommensurable, it holds equally good if the equisubmultiples taken of the first and third terms be infinitesimals. This question is considered fully in the Sixth and Seventh Notes of the Appendix; where, moreover, an illustration is given of Euclid's criterion which may make its meaning clearly understood by the student. The right conclusion as to the two tests is, probably, that both should be given in a treatise on elementary geometry, each having its own peculiar advantages.

To a few other matters I have to refer. The first is, that, as to strictness of demonstration and the bases on which it is made to rest, there is no geometric principle assumed in this treatise which has not in one form or another been assumed by Euclid. The introduction of Direction as a basis of the science takes the place of Euclid's fifth postulate, the difference being that the reality of direction is far more

evident than the postulate. The First Lemma here given is but little more than his assumption that "all right angles are equal;" and the Second is a larger and more definite account of the principle of "superposition." These are the only *geometric* assumptions here made—the first coming more properly under the head of a definition, and the other two of self-evident geometric theorems.

The Axioms I have separated from the Lemmas, for the reason that they are not purely *geometric* truths, but hold good as much in Algebra and Arithmetic as they do in Geometry. For this reason, I have described them as "principles of *quantitative* reasoning." A few new axioms are also added, as being wanted; and, into the statements of all, the word "magnitudes" has been introduced, to take the place of "things," as being more expressive and to the point.

As to the word "lemma," I am aware that it is not used here in strict accordance with its ordinary application; but it is a convenient term, and has been here adopted, agreeably to its derivation, solely to denote a pure geometric principle *received* as self-evident.

In the notation adopted, the student will find many advantages. It is applied according to the circumstances of the figure illustrating the demonstration; and, in any large class of cases, a uniformity of notation is as far as possible observed, as may be noticed in the Fifth Chapter. The impression on the student's mind at first might be that the numerals subscribed to the letters used in the demonstrations

would tend to produce confusion. He will find the result to be the very opposite, and that, moreover, after a little practice, he may read and understand the proof almost without looking at the figure.

As regards the demonstratious of the theorems and problems, it will be found throughout that a purity of geometric reasoning has been maintained free from the aid of Algebra, either in its methods or symbols. I have, indeed, frequently used such a symbol as $AO \cdot BO$ to denote a rectangle under two lines; but it is the only one, and withal legitimate. And, then, as to the general plan of the work, my aim has been simplification, for the reason assigned by Quetelet as to the teaching of the sciences:—"The principles of sciences rarely present attraction; but if the entrance to the edifice is not the most brilliant part, it ought, at least, to be easy of access and conveniently lighted."

W. A. W.

August, 1875.

CONTENTS.

AN INDEX

TO FIND EUCLID'S PROPOSITIONS.

The asterisks mark the propositions not given in this treatise.

ELEMENTARY GEOMETRY.

INTRODUCTION.

GEOMETRY is the Science of FORM in space subject to a law; and is either Plane or Solid.

A POINT is position in space without magnitude; and, as such, has been described as having "no parts."

A LINE is the path supposed to be described (whether by a law or irregularly) by the motion of a point, and is length without thickness.

DIRECTION* is an essential part of our conception of space, and also of a line in its supposed generation by the motion of a point. When the motion is in a straight line, there is always a *sameness* of direction; and when in a curved line, a continuous *change* of direction.

It is this sameness of direction which makes a right line "lie *evenly* between two points;" the evenness being *identity* of direction throughout.

LINES are therefore either Right Lines or Curves.

A RIGHT LINE is a line which throughout has the *same* direction.

---

* *See* Appendix, Note 1.

A CURVE is a line which throughout continuously *changes* its direction.

The only Lines we have to treat of in Elementary Geometry are the Right Line, and one Curve, the Circle.

THE RIGHT LINE is INDEFINITE or FINITE.

AN INDEFINITE RIGHT LINE extends in both directions to an infinite distance; and from its being marked throughout its course by its character of sameness of direction we term it "A DIRECTIVE."

To the FINITE RIGHT LINE we apply generally the term "LINE;" but sometimes "directive," when we treat only of its purely directive properties. No confusion can arise from this; as the only other line we have to treat of is the circle, which we shall always term "circle."

A PLANE is a surface, the directive through any two points of which lies on the surface.

Hence it follows that this surface can have no curvature, or bend, throughout; for, if there were a bend anywhere, the directive through two points on opposite sides of the bend could not lie throughout on the surface.

Hence, also, only one plane can pass through two intersecting directives; for, if two possibly could, the line joining any two points, one on each directive, should lie on both planes, which is impossible. This plane is *the* plane of the two directives.

In Elementary Geometry we treat only of figures on a plane, which is "the Plane of the Figures;" and all the definitions, theorems and problems here considered are about such plane figures.

A CIRCLE is a plane curve, every point of which is equally distant from another point in its plane, called "the centre."

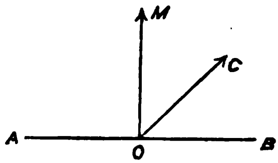
AN ANGLE is the DIVERGENCE OF TWO DIRECTIONS.

An angle is the "opening of two lines" or directives; and its magnitude in no way depends on the lengths of the lines which represent the directives. It is a species of space, quite distinct from *line*, *surface*, or *solid*, which may be termed "angular space."

Angles are either Right, Acute, Obtuse, or Re-entrant.

A RIGHT ANGLE is the angle which one directive makes with another when it meets it so that the two adjacent angles are equal.

If the directive OM intersect AB so that the angle AOM be equal to MOB, each of these angles are right angles.



AN ACUTE ANGLE is an angle less than a right angle.

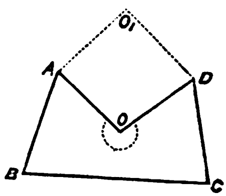
The directive OC makes an acute angle with OB.

AN OBTUSE ANGLE is an angle greater than one right angle but less than two.

The directive OC makes an obtuse angle with OA.

A RE-ENTRANT ANGLE is an angle greater than two right angles but less than four.

In the Figure the angle below the lines AO, DO, is a re-entrant angle, marked by the dotted arc, so called from the sides AO, DO, *going back*, as it were, into the figure from a position O_1, such as that represented by the meeting of the dotted lines AO_1, DO_1. It is one of the internal angles of the Figure, such as A is, but is evidently greater than two right angles.



THE SUPPLEMENT OF AN ANGLE is the angle which, with it, makes two right angles.

Thus two angles are said to be *supplementary* to each other when their sum is two right angles. In the first Figure above the angles AOC and BOC are supplementary.

THE COMPLEMENT OF AN ANGLE is the angle which, with it, makes one right angle.

Thus two angles are said to be *complementary* to each other when their sum is one right angle. In the first Figure above the angles BOC and MOC are complementary.

DIRECTIVES are either DIVERGENT or PARALLEL.

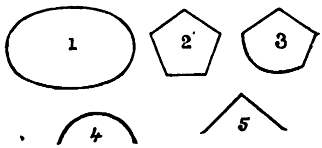
DIVERGENT DIRECTIVES are such as meet at some point, and thence diverge in *different* directions. They sometimes may be termed CONVERGENT directives.

PARALLEL DIRECTIVES are those which have the *same* direction.

LINES and CURVES, when they meet, form figures of various kinds, the terms used in reference to which should be known.

A PLANE FIGURE is a continuous outline in a plane formed of one or more lines, or curves, and is either *open* or *closed;* and is formed altogether of curves, or altogether of lines, or of a mixture of lines and curves.

Figs. 1, 2, and 3 are closed; 4 and 5 are open. Figs. 1 and 4 are altogether curved; 2 and 5 are made of portions of lines; and 3 is a mixture of lines and curves.



A closed figure always encloses an area, which an open figure cannot do. A circle is a closed figure, and encloses an area, which is to be carefully distinguished from the curve, which *alone* is *the* circle.

Closed Rectilineal Figures are, according to the number of their sides or angles, Triangles, Quadrilaterals, Pentagons, Hexagons, &c.; Polygon being a general term applicable to all.

A Triangle is a closed rectilineal figure formed of three lines; and is termed *Scalene* when the three sides are unequal. If two sides only be equal, the triangle is *Isosceles.* If all three sides are equal to each other, the triangle is *Equilateral.*

Triangles are named also in reference to their angles. If one angle be right, it is a *Right-angled Triangle;* if one be obtuse, it is an *Obtuse-angled Triangle;* and when all three angles are acute, it is an *Acute-angled Triangle.*

A Quadrilateral is a four-sided rectilineal figure, of which the chief varieties are the *Parallelogram,* the *Rectangle,* and the *Square.*

A Parallelogram is a Quadrilateral, the opposite sides of which are parallel.

A Rectangle is a Parallelogram, the angles of which are all right angles.

A Square is a Rectangle, the sides of which are all equal to each other.

THE AXIOMS;

OR, PRINCIPLES OF QUANTITATIVE REASONING.

1. Magnitudes which are equal to the same are equal to each other.

2. If equals be added to the same or equals, the wholes are equal.

3. If equals be taken from the same or equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Magnitudes which are doubles of the same, or of equals, are equal to each other.

7. Magnitudes which are the halves of the same, or of equals, are equal to each other.

8. The sum of the doubles of two magnitudes is double the sum of the magnitudes.

9. The difference of the doubles of two magnitudes is double the difference of the magnitudes.

10. The whole is greater than its part.

THE LEMMAS;*

OR, SELF-EVIDENT GEOMETRIC PROPOSITIONS.

1. The angular space round any one point is equal to the angular space round any other point.

2. Figures which can be so fitted to each other that their boundaries coincide all round, have their coinciding lines and angles equal, and, if they be closed, their enclosed areas also equal.

---

* The word "lemma" is derived from the Greek λαμβάνω, *to receive*, and means a proposition *received* as self-evident. There is another alleged derivation from λείπω.

THE PRINCIPLE OF DOUBLE CONVERSION.

If four magnitudes, a, b, A, B, are so related, that when a is greater than b, A is greater than B; and when a is equal to b, A is equal to B: then, *conversely*, when A is greater than B, a is greater than b; and, when A is equal to B, a is equal to b.

The truth of this principle, which extends to every kind of magnitude, is thus made evident:—If, when A is greater than B, a is not greater than b, it must be either less than or equal to b. But it cannot be less; for, if it were, A should, by the antecedent part of the proposition, be less than B, which is contrary to the supposition made. Nor can it be equal to b; for, in that case, A should be equal to B, also contrary to supposition. Since, therefore, a is neither less than nor equal to b, it remains that it must be greater than b.

NOTATION.

In this treatise such modes are adopted of denoting by letters the points, lines, angles, &c., which form the subjects of demonstration, as, in each case, seem best adapted to make evident the reasoning, and to diminish as far as possible the distraction of mind caused by continually alternating the attention between the steps of the argument and the lettering of the figures by which it is illustrated. A system of geometrical notation better than that in common use, and which would apply to every case, seems hardly attainable; but much may be done to render the old notation less perplexing than it usually is, at least in Elementary Geometry. As a general rule, therefore,

the old method is adhered to of denoting points by letters, lines by the letters representing any two points on them, and angles by the three letters which represent their vertices and any two points on their legs; and in like manner for more complex quantities, modifying it, however, with a view to clearness in the following particulars:—

1. Directives are denoted by the last letters (in capitals) of the alphabet, X, Y, Z, &c., and, in the figures, are represented with arrowheads affixed to one of their extremities, indicating their continuity beyond the bounds of the page and their proper directions. These letters are placed in advance of or beside the arrowheads, and are used in the course of demonstration generally to represent the directives; but in particular cases, where it tends to simplification, they are also used with two other letters, in the ordinary way, to indicate angles of which the directives are the legs.

2. Parallel directives are denoted by one of these last letters, annexing numerals to it to distinguish them from each other. For example, the letters X, X_1, X_2, X_3, represent four parallel directives, and Y, Y_1, Y_2, Y_3, four other and different parallels.

3. Whenever the case allows, without creating confusion, angles are denoted by the first letters of the Greek alphabet, α, β, γ, &c., writing them in their corners. Corresponding angles are denoted by the same letters with numeral suffixes.

4. As points often occur in doublets and triplets corresponding in some way to each other, instead of denoting them by different letters, the same letter is used with the numeral suffixes.

5. In cases of the comparison of two triangles, instead of denoting the vertices of one by different letters from those of the other, the same letters are used for both, the unit numeral being affixed to

those of the second. Thus, A, B, C, being the vertices of one triangle, A_1, B_1, C_1, are those of the other. And the angles of the triangles are indicated in a like manner, α, β, γ, for the one, and α_1, β_1, γ_1, for the other.

6. The sides of the triangle are denoted in the usual way by the letters corresponding to each pair of vertices. Thus AB and A_1B_1 indicate two corresponding sides in the two triangles.

7. A circle it is often found convenient to represent by only one letter R, denoting both the circle and its centre, describing them as "circle R" or "centre R." And when several circles have different centres, numeral suffixes are used. Thus, for example, R, R_1, R_2, &c., represent these circles and their centres.

8. A Parallelogram, rectangle or square, is notated generally, in the first instance, all round the figure by its vertices ABA_1B_1; but afterwards, in the course of demonstration, often the letters AA_1, at the extremities of a diagonal, are used to denote the parallelogram. The same two letters, or BB_1, are also used to denote the diagonal; but in all such cases we frequently use the contractions, the "parlgm." AA_1, or the "diagle." AA_1, as occasion requires. The advantage of this is apparent in the Fifth Chapter.

CHAPTER I.

THE DIRECTIVE.—THEOREMS.

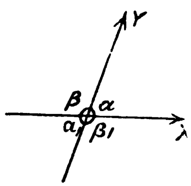
1. *Two Directives can intersect in only one Point.*

For, since the directives have different directions, they must, after meeting at one point, diverge from that point in their two different directions, and therefore not meet again.

This is equivalent to Euclid's 12th axiom, that "two right lines cannot enclose a space."

2. *The Opposite Angles of intersection of two Directives are equal.*

Let X and Y be the directives intersecting at the point O in the angles α, β, α_1, β_1. Then, since their directions in their convergence to O is the same as after their divergence from O, the difference of their directions before and after convergence must be the same; that is, the angles α and α_1 are equal; and β and β_1 are equal.



Or, as proved by Euclid, the sum of the angles α and β (*Def.*) is two right angles; and the sum of α_1 and β is also two right angles. Take away β from both sums, and α is equal to α_1. In like manner, it is proved that β is equal to β_1.

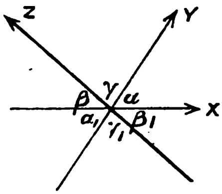
3. *All Right Angles are equal.*

If the two directives, X and Y, in the last Proposition, so intersect that the angles α and β be equal, then by Definition (*Introduction*) α and β are both right angles. But (*Theor.* 2)

α and β are equal to α_1 and β_1; and therefore the four angles α, β, α_1, β_1, are each a right angle. But each of these angles is the quarter of the angular space round the point O of meeting of the directives; and, since (*Lemma* 1) the angular space round any point is equal to that round any other point, all right angles, the quarters of that space, are equal.

4. *If three Directives meet in a point, and one of them makes angles on opposite sides with the other two, the Sum of which is two Right angles, the two Directives coincide.*

Let X, Y, and Z be the three directives; and let Y make, with X and Z, angles α and γ, the sum of which is two right angles. Then suppose (if possible), that X and Z do not coincide. Since α and γ together are, by supposition, two right angles, the sum of the opposite angles α_1 and γ_1 are (*Theor.* 2) also two right angles. Add to both sums respectively the angles β and β_1; and the sum of the six angles about the meeting of the three directives is more than four right angles; which is impossible, since the angular space round any point (*Theor.* 2) is only four right angles. The directives X and Z must, therefore, coincide.

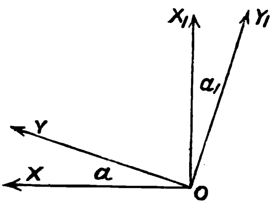


5. *Parallel Directives cannot meet.*

For, were they to meet at any point, after meeting they should diverge from that point, and have different directions; which is impossible, since, by definition, they have the same direction.

6. *The Angle between two Directives is equal to the angle between any two Directives Perpendicular to them.*

Let X and Y be the two directives, and X_1, Y_1, two others perpendicular to them at their point of divergence, O. Then, since the angles XOX_1 and YOY_1 are right angles, on taking from both the angle YOX_1, the remaining angles α and α_1 are equal, that is, the angle between X and Y is equal to that between X_1 and Y_1.



If the two perpendicular directives do not meet at O, but at any other point, the result is the same; for the angle between them

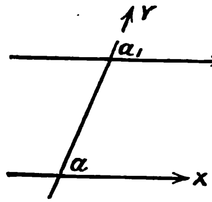
is the same as that between X_1 and Y_1 parallel to them, their directions being the same, and their divergences therefore equal.

7. *Only one Parallel to a Directive can be drawn through a Point.*

For, if two could be drawn, each would have the same direction as that of the directive. But, by supposition, these two parallels pass through the point. They must therefore diverge from that point, and have different directions; and therefore not be both parallels to the directive.

8. *The Angles of intersection of a Transversal with two Parallel Directives are equal.*

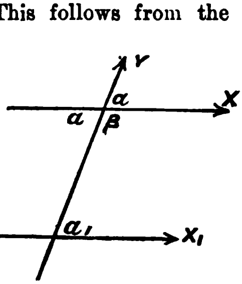
Let X, X_1 be the parallel directives, and Y the transversal cutting them at the angles a, a_1. Then, since X, X_1, being parallel, have the same direction, and the direction of Y is the same throughout its extension, the divergences of Y from X and X_1 are the same and equal. But these divergences are the angles a and a_1, which are therefore equal.



9. *A Transversal cuts two Parallel Directives so that the Alternate Angles are equal, and that the Sum of the two Internal Angles on either side are equal to two Right Angles.*

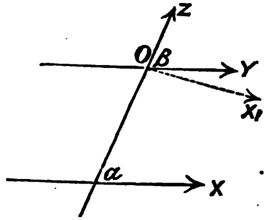
This follows from the preceding Proposition. Since α_1 is equal to α, as proved above; and (*Theor.* 2) α is equal to the α vertically opposite to it; the alternate angles α_1 and α are equal.

Also; since α is equal to α_1, and the Sum of α and β is two right angles, the Sum of the internal angles α_1 and β is also two right angles.



10. *If a Transversal cut two Directives and make the angles of intersection with them equal, the directives are Parallel.*

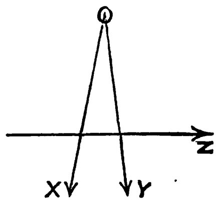
Let X and Y be the two directives, and Z the transversal; and, if possible, suppose that Y is not parallel to X. Then some other line, OX_1 through O, is parallel to X. The angle ZOX_1 is therefore (*Theor.* 8) equal to α. But, by supposition, the angle β, or angle ZOY, is equal to α; and therefore β and ZOX_1 are equal, a part equal to the whole, which (*Axiom* 10) is impossible, unless the directives X_1 and Y coincide. Therefore X and Y are parallel.



COR. 1. Hence it is evident that, if the transversal Z were to cut X and Y so that the alternate angles were equal, or the sum of the two internal angles on either side were equal to two right angles, the same conclusion would hold, viz., that the directives X and Y are parallel.

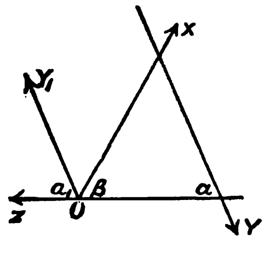
11. *Two Perpendiculars cannot be drawn from a Point to a Directive.*

For suppose there could; and that X and Y were two directives from a point O, both perpendicular to the directive Z. Then, since Z makes equal angles, each a right angle, with the directives X and Y (*Theor.* 10) these directives are parallel. But, by supposition, X and Y meet at O; and therefore diverge from O; they cannot therefore be parallel; and therefore not both be perpendicular to Z.



12. *The Angles of intersection of a Transversal with two Intersecting Directives are unequal, the External greater than the Internal; and the sum of the two Internal Angles less than two right angles.*

Let X and Y be the directives, and Z the transversal cutting them at the external angle ZOX and the internal α between Y and Z. From the point O let Y_1 be a parallel to Y. Then, since Y and Y_1 are cut by the transversal Z, the angles α and α_1 are equal, and therefore the angle ZOX, which is greater than α_1, is greater than the internal angle α; *as stated.*



Also, since the angle ZOX is greater than α, and ZOX with β is two right angles, the sum of the internal angles α and β is less than two right angles, *as stated.*

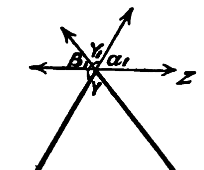
13. *If a Transversal meet two Directives, and make angles with them, the External greater than the Internal, or the sum of the two Internal Angles less than two right angles, the two directives must meet.*

For, suppose they do not meet. Then, they should be parallel; and the external angle should be equal (*Theor.* 8 & 9) to the internal, and the sum of the two internal angles should be equal to two right angles; which is contrary to the supposition made; and therefore the directives must meet, *as stated.*

Since a triangle is formed of three *finite* portions of directives, it is evident that the relations of the angles of a triangle are those of three intersecting directives. Hence follows the statement of the angular properties of a triangle.

14. *The Sum of the Internal Angles of a Triangle is equal to two right angles.*

Let ABC be the triangle, and α, β, γ, the three internal angles; and let Z be a directive through the vertex C parallel to AB. Then, since Z is parallel to AB, the angle of intersection α_1 of AC with Z is equal to the angle α with AB. Also, the angle of intersection β_1 of BC with Z is equal to the angle β with BA. The angles vertically opposite γ and γ_1 (*Theor.* 2) are also equal. Hence the sum of the angles α, β and γ is equal to the sum of α_1, β_1 and γ_1; but this latter sum is evidently two right angles. Therefore, the sum of the three angles is two right angles, *as stated.*



COR. 1. Hence, the external angle of a triangle is equal to the sum of the two internal opposite angles.

For, the external angle, together with its adjoining internal angle, is equal to two right angles. But the three internal angles (as now proved) are also equal to two right angles. Take away the adjoining internal angle from both sums, and the external angle is equal to the sum of the two internal opposite angles.

COR. 2. Hence, also, the external angle is greater than either of the internal opposite angles; as has been already (*Theor.* 12) proved.

COR. 3. A triangle cannot have more than one obtuse or one right angle.

For, if it had, the two obtuse or two right angles with the third angle would together be greater than two right angles, which by the above theorem is impossible.

COR. 4. If one angle of a triangle is right, the other two are acute, and each the complement of the other.

For, one being right, the other two together are equal to one right angle; and therefore each is the complement of the other and acute.

COR. 5. If two angles of a triangle are equal, they are each an acute angle.

For, if they were both obtuse or right angles, the sum of the three angles of the triangle would be greater than two right angles, which is impossible.

COR. 6. If the three angles of a triangle are equal to each other, they are each two-thirds of a right angle.

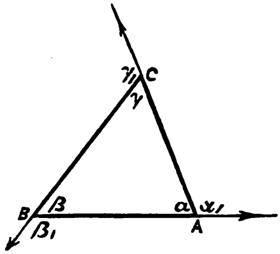
For, each in that case is the third of two right angles, that is, two-thirds of one right angle.

COR. 7. If the two base angles of a right-angled triangle are equal, they are each equal to half a right angle.

For, since their sum is one right angle and they are equal, each must be half a right angle.

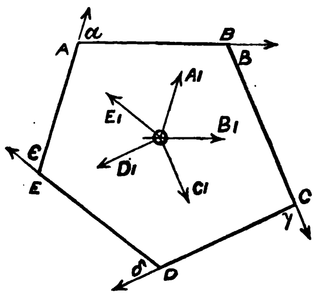
15. *The Sum of the External Angles of a Triangle is equal to four right angles.*

For, since each external angle with its corresponding internal together make two right angles, the sum of the six angles, three internal and three external, is equal to six right angles. But (*last Theor.*) the sum of the three internal angles α, β, γ, is equal to two right angles. Take these angles away from the six right angles, and the sum of the three external angles α_1, β_1, γ_1, is equal to the remaining four right angles.



16. *The Sum of the External Angles of a Polygon is equal to four right angles.*

From a point, O, anywhere placed within the polygon, draw lines OA_1, OB_1, OC_1, &c., parallel to the sides of the polygon. Then, since OA_1 is parallel to EA, and OB_1 to AB, the angle A_1OB_1 is (*by Parallelism*) equal to the angle α. In like manner, since OB_1 and OC_1 are parallel respectively to AB and BC, the angle B_1OC_1 is equal to the angle β. And so on, going round the figure until we return to A, it can be shown that the angles round O are equal respectively to the external angles α, β, γ, δ, and ϵ of the polygon. But the angles round O are four right angles. Therefore the sum of the external angles is equal to four right angles.



17. *The Sum of the Internal Angles of a Polygon is equal to the Excess of double as many right angles as the polygon has sides over four right angles.*

As each internal angle with its corresponding external is equal to two right angles, the sum of all the internal and all the external is equal to twice as many right angles as the polygon has angles, or sides. If we take from this latter sum the four right angles to which the external angles (*last Theor.*) are proved

equal, the remainder, the sum of the internal angles, is equal to double as many right angles as the polygon has sides less by four right angles.

COR. 1. In a regular polygon, or in one all the angles of which are equal, each external angle is the angle that would be obtained by dividing four right angles into as many equal parts as the polygon has sides.

COR. 2. Each internal angle of a regular polygon is the supplement of the angle which would be obtained by dividing four right angles into as many equal parts as the figure has sides.

COR. 3. The angle of a regular or equiangular pentagon is the fifth part of six right angles; that of a regular hexagon the sixth part of eight; of a regular heptagon the seventh part of ten; of a regular octagon the eighth part of twelve; and so on for other regular polygons.

CHAPTER II.

THE CIRCLE.—THEOREMS.

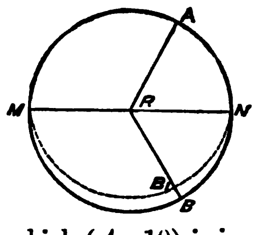
THE properties of the directive derived from its definition being established, the next are those of the circle.

From definition the circle ought to be uniform in shape and curvature. Every point is equidistant from the centre; and the mode of describing any one part being the same as that for any other part, the form in every part ought to be the same. Therefore any oval forms, or forms semiconvex-semiconcave, are really inadmissible, and can in no way represent a circle. But, in order to strictness of demonstration, it is necessary to establish these presumptions by direct reasoning.

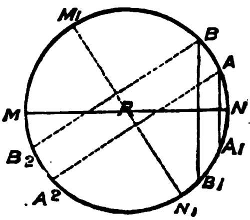
The principle by which this may be done, is that of *superposition*, supposed to take place but not necessarily actually performed. The principle was adopted by Euclid in the fourth proposition of his First Book in the superposition of lines and angles, and in his Third Book for segments of circles; but we apply it in the form of a supposed superposition of arcs and lines.

It he portion of a circle on one side of a diameter be supposed turned round that diameter so that it may fall on the other side, the two portions must throughout cover each other, or coincide.

For, suppose MN to be the diameter, and that the upper portion turned round MN does not cover the lower, but becomes the dotted curve. Then, since the dotted curve MB_1N is the same as the curve MAN, the radius RB_1 is equal to RA. But RB is also equal to RA, both being radii. Therefore RB_1 is equal to RB, a part equal to the whole, which (*Ax.* 10) is impossible. The two portions then must coincide.

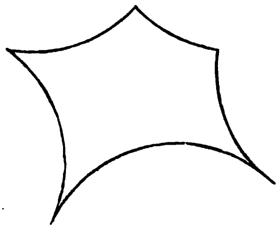


But, as the superposition may be made round any diameter, any assigned small arc may be made to coincide with any number of equal small arcs all round the circle. For, AB being the assigned arc, and A_1B_1 any other equal arc; then, drawing a diameter MN through the middle point of the arc AA_1, we have the arcs AN and A_1N equal. Adding to both the equal arcs AB and A_1B_1, the arcs BN and B_1N are equal. Now, on turning the upper semicircle round MN so that it fall on the lower one, the two will coincide. But, since arc AN is equal to arc A_1N, the points A, A_1, coincide. Also, since arc BN is equal to arc B_1N, the points B, B_1 coincide. But, since the two semicircumferences also coincide and cover each other, the arcs AB and A_1B_1 also coincide throughout their lengths.



The same may be done for any other arc A_2B_2 by taking the diameter M_1N_1 so that it bisect the arc BB_2, and AB may be made to coincide with A_2B_2; and so on for any number of arcs. Whence, we conclude that the circle has the same form or curvature all round, wholly concave to the centre or wholly

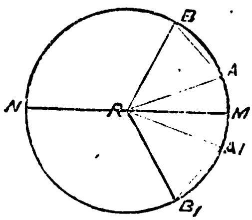
convex to the centre, but in no case partly concave and partly convex.



But it cannot be wholly convex to the centre. Such a closed figure cannot be imagined without having angular breaks (as in the Figure), which cannot possibly represent a curve of the *same form* throughout.

These principles being established as to the outline and form of the circle, its distinctive properties follow.

1. *Equal Arcs of a Circle subtend Equal Angles at the centre, and have Equal Chords.*

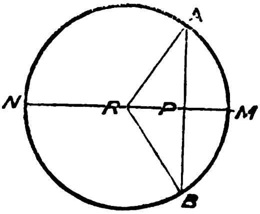


In the introduction to this chapter it has been proved that the arc AB can be made to coincide with the equal arc A_1B_1. In the coincidence the radii RA and RB coincide with RA_1 and RB_1. The angles, therefore, between RA and RB, RA_1 and RB_1 are equal, and therefore the arcs subtend equal angles at the centre. Also; since the extremities of the arcs coincide, the *lines* AB A_1B_1, coincide, and are equal, that is, the chords of equal arcs of a circle are equal.

COR. 1. This holds for Equal Circles; for all equal circles by superposition may be made to coincide and become one circle.

2. *The Perpendicular from the Centre of a Circle on a chord bisects the Chord, and also bisects the Arc and the Angle it subtends at the centre; and the chord makes Equal Angles with the radii to its extremities.*

Let AMB be the arc, and AB the chord, and $NRPM$ the perpendicular from the centre R on AB produced into a diameter. If we suppose the semicircle NAM turned round the diameter NM and placed below NM, it will then coincide with the semi-circle NBM. But in this superposition AP being at right angles to RP will turn round NM so as to cover BP; and the point A will coincide with B.



1. The lines AP and BP are therefore equal; *or the chord AB is bisected.*

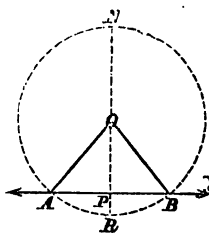
2. But further, since A coincides with B, the arc AM is equal to BM; and therefore, *the arc AB is bisected by the perpendicular.*

3. Also, since the arcs AM and BM are equal (*Theor.* 1), the angles at the centre ARM and BRM are equal; that is, *the perpendicular on the chord bisects the angle at the centre subtended by the chord.*

4. Since RA and PA in the superposition coincide with RB and PB respectively, the angles RAP and RBP are equal; *or the chord makes equal angles with the radii through its extremities.*

3. *A Directive can cut a Circle in only two Points.*

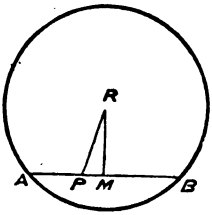
For, let X be the directive cutting the circle OR in A and B, and NR a diameter perpendicular to X, or AB. Then, on turning the semicircle NAR round NR so that it fall on NBR, if there were three *distinct* points in which X could cut the circle, two of them should be on one side, suppose NBR, of the circle, and the other point A could not on superposition possibly coincide with both; and therefore there are only two points of intersection.



If possibly four points of intersection could be supposed, then the circle should be partly convex and partly concave, which we have shown to be impossible, as proved above.

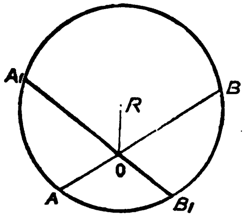
4. *A line drawn from the Centre of a Circle to the Bisection of a Chord is perpendicular to the chord.*

Let AB be the chord, and RM the line from the centre to its bisection. Then, if RM be not perpendicular to AB, some other line RP from the centre is perpendicular. But since RP is perpendicular to AB (*Theor.* 2), it bisects AB in P. Therefore the line AB is bisected twice, at M and at P; which is impossible; since then PA should be equal to MA, a part to the whole. (*Ax.* 10.)



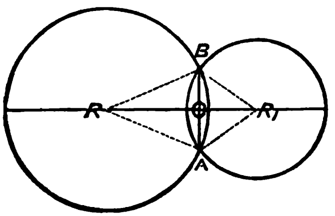
5. *Two Chords intersecting within a Circle at a point not the centre cannot bisect each other.*

For, suppose they could bisect each other, and AB and A_1B_1 were the bisected chords. Then, the line from the centre R to their intersection O (*last Theor.*) should be perpendicular to both AB and A_1B_1, which is impossible; since AB and A_1B_1 are lines having different directions, and could not make equal angles, each a right angle, with the line RO from the centre, for, in that case, a part would be equal to the whole. (*Ax.* 10.)



6. *The Line joining the Centres of Two Intersecting Circles bisects their common chord, and is perpendicular to it; and also bisects the angles subtended at the centres by the common chord.*

Let AB be the common chord of intersection of the circles, R, R_1, their centres, and O the bisection of AB. Suppose R and R_1 joined with O. Then, since the chord AB of circle R is bisected at O, the line RO is (*Theor.* 4) perpendicular to AB. In like manner, AB being a chord of circle R_1, the line R_1O is also perpendicular to AB. These perpendiculars through O make therefore one continuous line passing through the centres R,R_1. Therefore RR_1 bisects the common chord AB.



The line RR_1 also bisects the angles ARB and AR_1B; for (*Theor.* 2) the perpendiculars RO and R_1O bisect the angles ARB, AR_1B, subtended at the centres R, R_1 by the chord AB.

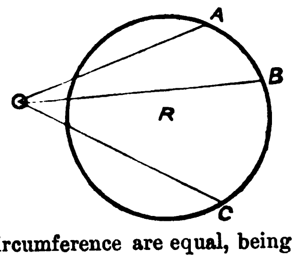
7. *Two Circles cannot intersect in more than Two Points.*

For, if possible, suppose them to intersect in three points. Then, the two circles should have three common chords; and (*by last Theor.*) the line joining their centres should be perpendicular to each of these common chords; which is impossible, since the three chords form the sides of a triangle, and no one line can be perpendicular to all three.

Were the circles to intersect in four points, the difficulty would be increased, as there would then be six common chords, the four sides and two diagonals of a quadrilateral; and the line joining the centres could much less be perpendicular to all six.

8. *From a Point, not the centre of a circle, more than Two Equal Lines cannot be drawn to its circumference.*

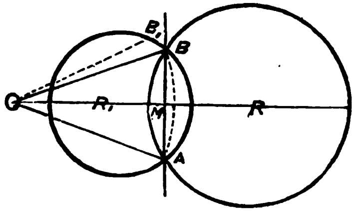
For, suppose there could, and that O were the point, and OA, OB, and OC, three equal lines to the circle R. Then, if, with O as centre and one of the lines OA as radius, a circle be supposed described, this circle should cut the circle R in three points, A, B, and C; but (*by last Theor.*) this is impossible. Only two equal lines can therefore be drawn, *as stated.*



The centre is excepted, because all lines through it to the circumference are equal, being radii.

9. *Two Lines drawn from any point on the directive, joining the centres of Two Intersecting Circles to the extremities of their Common Chord, are equal, and make equal angles with the joining directive.*

Let R and R_1 be the two circles intersecting in the common chord AB, and O the point from which the lines OA, OB are drawn to A and B. Then, if the lines OA and OB are not equal, but OA is greater than OB, a circle, with O as centre and OA as radius, must pass through some point B_1 on AB produced; and therefore the chord of that circle, AB_1, is bisected at M (*Theor.* 2) by the perpendicular OM from O its centre. But the common chord AB of the circles R and R_1 is also bisected by OMR; and therefore AM is equal to both BM and B_1M, and BM is equal to B_1M, a part to the whole (*Ax.* 10). The lines OA and OB are therefore equal; and, in the circle OAB, make equal angles with the line joining the centres OR_1R.



N.B.—This proposition proves that the directive RR_1 is the *locus* of the centres of all circles having AB for a common chord.

10. *If from Any Point any number of pairs of Equal Lines be drawn to a circle, these pairs of lines make equal angles respectively with the Diametral Line through the point on each side of it.*

Let O be the point, and (Fig. 1) R the circle in the diagram, to which the pairs of equal lines OA, OA_1, OB, OB_1, &c., are drawn, and $MORN$ the diametral line. Then, since the pairs of lines are equal, circles can be described with O as centre, and OA, OB, OC, as radii passing through A_1, B_1, C_1, and the lines connecting AA_1, BB_1, CC_1, will be the common chords of these circles and the given circle. But the common chord (*Theor.* 6) of two intersecting circles subtends at the centre of either circle an angle which is bisected by the line joining their centres. Therefore the angles AOA_1, BOB_1, COC_1, &c., are bisected by MON, the diametral line; and the equal lines therefore make equal angles with MON, *as stated.*

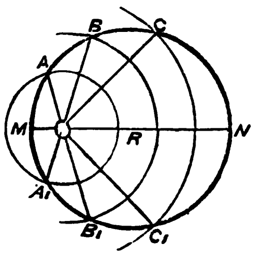


Fig. 1.

When the point is without, or on the circle, the same proof holds good; the position of the point O only changes. But the lines remain the same as regards the demonstration, as illustrated by Figs. 2 and 3.

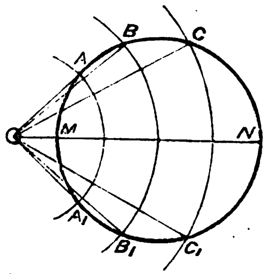


Fig. 2.

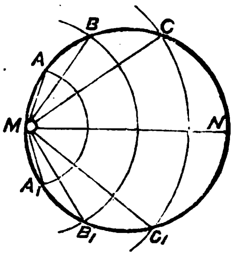
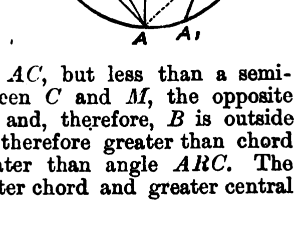


Fig. 3.

11. *The Greater of Two Arcs of a circle, neither greater than a Semicircle, has the greater chord, and subtends the greater angle at the centre.*

Let AB and A_1B_1 be the arcs, and suppose the arc AC taken equal to arc A_1B_1. Then, since (*Theor.* 1) the arcs AC and A_1B_1 being equal, have equal chords and subtend equal angles at the centre, it is sufficient to prove that arc AB has a greater chord and subtends a greater angle at the centre R than the arc AC. Suppose then, a circle described with A as centre and AC as radius; this circle meets the circle R in only one other point, C. But, the



arc AB being greater than arc AC, but less than a semicircle, its extremity B lies between C and M, the opposite extremity of the diameter ARM; and, therefore, B is outside the circle ACC. Its chord AB is therefore greater than chord AC; and the angle ARB is greater than angle ARC. The greater arc therefore has the greater chord and greater central angle, *as stated.*

N.B.—The limitation in this proposition, that the arcs should be less than a semicircle, depends on the circumstance that,

when an arc exceeds a semicircle, thence up to a whole circumference, the greater arc has the lesser chord. Thus the arc $ACBMB_1C$ has a chord AC less than the chord of ACB.

The angles at the centre are, however, always greater for the greater arc; but, after the arc exceeds a semicircle, the central angle becomes *re-entrant.*

N.B.—This holds good for chords of equal circles; for all equal circles may by superposition be made to coincide and become one circle.

Hence, by applying the Principle of Double Conversion (*Introduction*), it follows that, since (*Theor.* 1) when two arcs of a circle are equal, their chords and angles at the centre are equal; and since, also (*as here proved*), that when one arc (less than a semicircle) is greater than another, the greater arc has the greater chord, and subtends the greater angle at the centre; so:—

Conversely, Equal Chords of a circle have equal arcs; and the Greater Chords greater arcs; and equal angles at the centre are subtended by equal arcs, and greater angles by greater arcs.

This conclusion holds good for equal circles.

12. *The Angle subtended at the Centre by an Arc of a Circle is double the angle subtended by it at the Remaining Circumference.*

When the angle at the circumference is acute there are three cases.

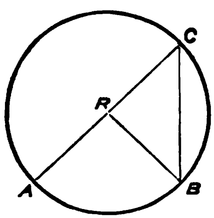


Fig. 1.

1. When one of the legs of the angle at the circumference passes through the centre.

2. When the centre lies between the legs of the angle.

3. When the centre lies outside both legs.

First Case.—Let the arc be AB and ACB the angle subtended at the circumference, and ARC the leg through the centre R. Then, since CB is a chord, it makes (*Theor.* 2), equal an-

gles RBC, RCB with the radii RB, RC. But, the angle ARB being external, is (*Theor.* 14, *Cor.* 1, *Chap. I.*) equal to the sum of the equal internal angles, RBC and RCB; or to double ACB, *as stated.*

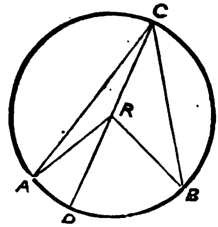


Fig. 2.

Second Case.—The centre being within the angle ACB, let CRD be the diameter through C, and RA, RB, the lines from the extremities of the arc to the centre. Then, as has been proved in the first case, the angle ARD is double of the angle ACD, and also the angle BRD double the angle BCD. Therefore the sum of ARD and BRD (*Ax.* 8) is double the sum of ACD and BCD, that is, double ACB, the angle at the circumference, *as stated.*

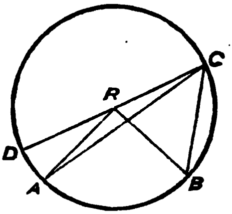


Fig. 3.

Third Case.—The centre being outside the angle ACB, let CRD be the diameter through C, as before. Then, as in the first case, the angle BRD is double BCD, and also ARD double ACD; therefore the difference of BRD and ARD is (*Ax.* 9) double the difference of BCD and ACD, that is, double ACB, *as stated.*

In the diagrams of the above cases the angle ARB at the centre is less than two right angles, those at the circumferences being acute, and the arc AB less than a semicircle. But if, in the second Figure, we place AR and BR *in directum*, ARB becomes a diameter, and the angle ARB becomes two right angles, and therefore the angle ACB is *one* right angle. This is further evident from the fourth Figure. As proved in the first case above, the angle ARD is double ACD; and also BRD double of BCD. Therefore the sum of ARD and BRD is double the sum of ACD and BCD, or double ACB. But the sum of ARD and BRD is two right angles. Hence :—

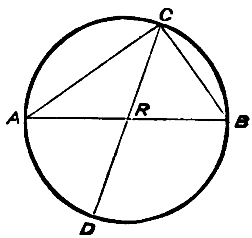


Fig. 4.

The angle in a semicircle is a right angle.

If the arc AB become greater than a semicircle, the angle at

the centre, R, is the re-entrant angle ARB, and therefore the angle ACB standing on AB is obtuse, as is evident from Figure 5. For the angles ARD and BRD (*first case above*) are the doubles respectively of ACD and BCD. But the sum of ARD and BRD is greater than two right angles. Therefore ACB is greater than one right angle, that is, ACB is obtuse. Hence:—

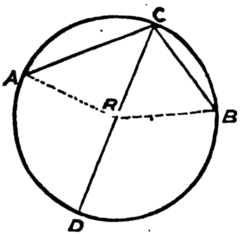


Fig. 5.

The angle in a segment less than a semicircle is Obtuse.

From the first three cases, the angles at the centre, being less than two right angles, the arc ACB is greater than a semicircle, and the angles at the circumference are acute. Hence:—

The angle in a segment greater than a semicircle is Acute.

These three theorems may be expressed otherwise:—

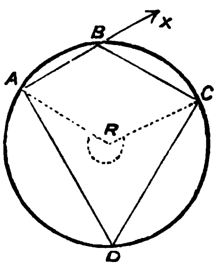
1. *An angle standing on a semicircle with its vertex on the remaining semicircle is a Right angle.*
2. *An angle standing on an arc greater than a semicircle with its vertex on the remaining circumference is Obtuse.*
3. *An angle standing on an arc less than a semicircle with its vertex on the remaining circumference is Acute.*

13. *All angles in the Same Segment of a circle, or standing on the Same Arc of the circle with its vertex on the Remaining Circumference, are equal.*

For all such angles are the halves of the angles at the centre—acute, obtuse, two right angles or re-entrant, as has been proved in the *last theorem.*

14. *The sum of the Opposite Angles of a Quadrilateral inscribed in a circle is two right angles.*

Let $ABCD$ be the quadrilateral in the circle R; and suppose the centre R joined with A and C. Then the angle ARC at the centre R (*Theor.* 12) is double of the angle ADC. Likewise the re-entrant angle ARC is (*same Theor.*) double of the angle ABC. But ABC and its re-entrant ARC is equal to four right angles. Therefore the sum of their halves, ADC and ABC, is equal to two right angles, *as stated.*



Cor. 1. It is evident that the angle XBC, being the supplement of

ABC, is equal to the internal opposite angle ADC of the quadrilateral.

THE TANGENT TO THE CIRCLE.

It having been proved (*Theor.* 2) that the diameter of a circle perpendicular to a chord bisects it, it follows that, as the chord moves parallel to itself outwards towards the extremity of the diameter, the two semichords tend each to become *zero* together, while they still remain equal. On reaching the extremity both semichords vanish, while the moving line remains perpendicular. In this position the line becomes a tangent, neither cutting the circle nor yet being separated from it. Such is the tangent—a line perpendicular to the radius at the point of contact.

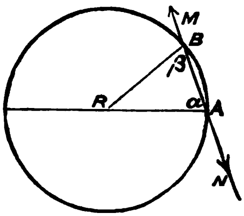
The perpendicularity of the tangent to the diameter may also be inferred, from the definition of a circle and the mode of describing one. The point of one leg of a compass being at the centre, the motion of that of the other must be always perpendicular to the line joining the extremities of the legs, that is, to the radius. But the direction of the motion of the generating point of any curve is that of the tangent to the curve at that point. It therefore follows that the tangent to a circle is perpendicular to the radius through the point of contact.

It is also evident that, the circle being a simple curve, there can be only *one* tangent to it at any point.

The following demonstration from EUCLID, slightly altered, also shows that the line perpendicular to the diameter is the tangent.

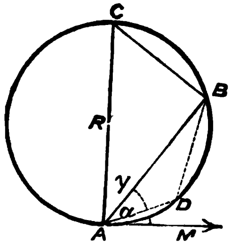
15. *The Line at an Extremity of a Diameter of a circle Perpendicular to it falls without the circle, that is, is a tangent to the circle.*

For suppose it do not fall without the circle, but cuts it. Let MN be the line through A cutting the circle in another point, B. Then on joining the centre R with A and B, the chord AB (*Theor.* 2) makes equal angles, α, β, with the radii RA, RB. But, by supposition, α is a right angle. Therefore β is also a right angle; and the sum of the two angles α, β, of the triangle ARB, is two right angles, which is impossible (*Theor.* 12, *Chap. I.*). Therefore MN does not again cut the circle, and is *the* tangent, *as stated.*



16. *The Angle between a Tangent to a circle and a Chord through the point of contact is equal to the angle in the alternate segment.*

Let AM be the tangent at A in the circle R, and AB the chord, making any angle, α, with AM, and also AC the diameter through A. Then, since CAM is a right angle (*last Theor.*), the sum of α and γ is a right angle. Also, since the arc ABC is a semicircle, the angle ABC (*Theor.* 12) is right; and the sum of the angles C and γ is therefore a right angle. Take γ from the two sums, and α is equal to C, the angle in the alternate segment ABC, *as stated.*



17. *Two Circles which have the Same Centre cannot cut or touch each other.*

For the radii of the circles are either equal or unequal. If they be equal, the circles coincide throughout, and are virtually one circle. If the radii be unequal, they must lie one within the other; and therefore can neither cut nor touch.

CONVERSELY.—*Two intersecting or touching circles cannot have the same centre.* For, supposing them to have the same centre, their radii are either equal or unequal. If they be equal, they coincide and can neither cut nor touch. If they be unequal, they lie one within the other, and likewise can neither cut nor touch. They, therefore, have not the same centre, *as stated.*

18. *The Line joining the centres of two Touching Circles passes through a Point of Contact of the circles.*

There are two cases, External contact, and Internal.

First Case. — External contact. Let R and R_1 be the circles, and A a point of contact, and MN the tangent common to the circles at A. Then RA (*Theor.* 15) is perpendicular to MN at A. So, likewise, is R_1A perpendicular to MN at A. The two lines RA and R_1A (Fig. 1) being thus perpendicular to MN at the same point, A, are *in directum*, and form one line RR_1. Therefore the line RR_1, joining the centres, passes through that point of contact.

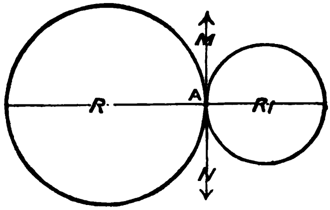


Fig. 1.

Second Case.—Internal contact. The reasoning is the same as above. The lines RA and R_1A (Fig. 2) are both perpendicular to MN at A; and therefore form one line RR_1, the production, RA, of which passes through a point of contact of the circles.

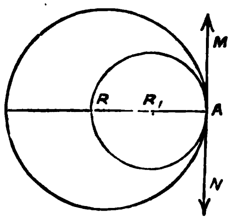


Fig. 2.

19. *Two Circles can touch in only One Point.*

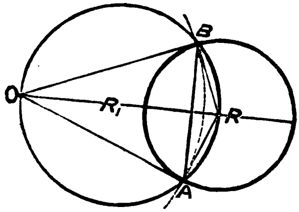
For, using the above figures, we have the two cases again of external and internal contact.

First Case.—External. If there be a second contact, the line RR_1 should pass also through it (*by last Theor.*). But that can only take place at the extremities of the diameters on opposite sides of MN. But this, in external contact, is impossible; for these extremities are on the opposite productions of the finite line RR_1, and the circles cannot possibly touch or meet at these extremities.

Second Case.—Internal. Here, as in the first case, the second contact can only take place at the extremities of the diameters to left of A. But in that case (Fig. 2) the diameter of the internal circles should be equal to that of the external; and the two circles should have equal radii and coincide, not touch.

20. *The Two Tangents from a Point to a Circle are equal, and make equal angles with the Diametral Line joining the point with the centre of the circle.*

Let the circle R be that to which the two tangents OA, OB, are supposed drawn from the point O. Then, since RAO and RBO (*Theor.* 15) are right angles, the points A and B are on a circle (*Theor.* 12) having OR for diameter, the bisection R_1 of OR for centre, and AB a chord common to it and the circle R. But (*Theor.* 9) the lines OA, OB, drawn from a point O anywhere on a directive RR_1, joining the centres of two intersecting circles,



R, R_1, to the extremities of their common chord, are equal and make equal angles with that directive. Therefore the tangents OA, OB are equal; and the angles AOR and BOR are also equal, *as stated.*

COR.—The chord AB is evidently also bisected by OR, and is perpendicular to OR.

PROBLEMS.*

The geometric constructions by which problems are solved rest ultimately on three simple operations, which are taken for granted as elementary. Hence the statements of the legitimacy of these operations are called "postulates"—a term which means that the geometrician postulates, or asks that these elementary constructions be allowed him. The postulates amount to a permission to use the Ruler for the purpose of connecting two points or extending a finite line, and the Compass for describing a circle with any given line as radius and one of its extremities as centre. They are generally given with the Definitions and Axioms; but their proper place is in this chapter at the head of the first group of problems.

POSTULATES.

1. A Right Line may be drawn connecting any two points.
2. A Finite Right Line may be produced through either of its extremities.
3. A Circle may be described with any point as centre, and any line for radius of which that point is one extremity.

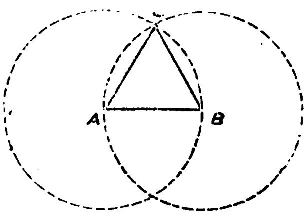
---

1. *On a given finite line AB to construct an equilateral triangle.*

With A as centre and AB as radius describe a circle. With B as centre and with the same radius also describe a circle.

* *See* Appendix, Note 2.

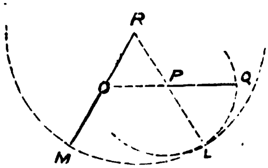
These circles must meet, since the centre of each is within the other. Let C be one of their points of meeting; join C with the points A, B. The triangle ACB is the required equilateral triangle.



For, since AC and BC, being radii of their respective circles, are each equal to the common radius AB, they are (*Ax.* 1) equal to each other. Therefore the three sides of the triangle are equal, that is, the triangle is equilateral.

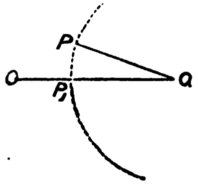
2. *From a given point O to draw a line equal to a given finite right line PQ.*

First Case.—Let the point O be on the line PQ or its production. Then on OP construct (*Prob.* 1) an equilateral triangle ORP. Next, with P as centre and PQ as radius, describe a circle cutting RP produced in L. Again, with R as centre and RL as radius, describe another circle cutting RO produced in M. Then OM is the required line.



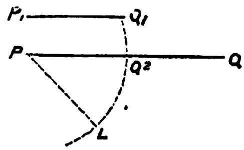
For, in the circle described with R as centre and RL as radius, the lines RL and RM are equal, being radii. From these equals take away respectively the equal sides RP and RO of the equilateral OPR, and the remainders (*Ax.* 3) OM and PL are equal. But PL is equal to PQ, both being radii of the circle described with P as centre. Therefore OM is equal to PQ, and is the *required* line.

Second Case.—If the point O be not on PQ or its production, then join O with either extremity Q of PQ; and with Q as centre and PQ as radius describe a circle cutting OQ in P_1. Then the problem is reduced to the first case, O being on the continuation of P_1Q, and P_1Q being equal to PQ. If, therefore, a line be drawn (*First Case*) through O equal to P_1Q, it will be equal to PQ.



3. *From the greater of two unequal lines PQ, P_1Q_1, to cut off a part equal to the less.*

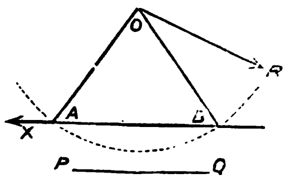
Draw a line PL (*Prob.* 2) from the extremity P of the greater line PQ equal to the less P_1Q_1, and with P as centre, and this line as radius, describe a circle cutting PQ in Q_2. The portion PQ_2 is evidently the required segment.



4. *To draw a line of a given magnitude PQ from a point O to a directive X.*

From the point O draw (*Prob.* 2) a line OR equal to PQ. With O as centre and OR as radius, describe a circle cutting the directive X in A and B. Either of the two lines, OA, OB, from O to the two points of intersection is the required line equal to PQ. This is evident from the construction.

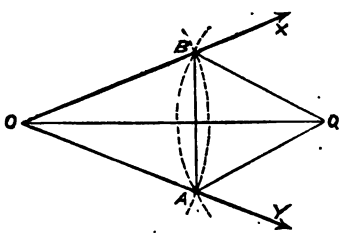
If PQ be less than the perpendicular from O on the directive, the problem is impossible.



5. *To bisect the angle between two directives, X, Y.*

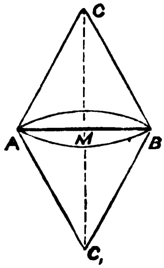
Let O be the point of divergence of the directives. With O as centre and any distance OA on Y, let a circle be described cutting X in B. Join AB, and construct on it an equilateral triangle AQB. Join Q with O. The line QO bisects the angle XOY.

For, Q, being the vertex of an equilateral triangle AQB, is also the centre of a circle passing through the points A, B, and having a common chord AB with the circle AOB. The line QO joining their centres (*Theor.* 6) therefore bisects the angle XOY, *as required.*



6. *To bisect a given line AB.*

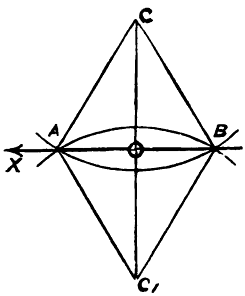
On both sides of AB construct equilateral triangles ACB, AC_1B. Join their vertices CC_1. The joining line bisects AB at M.



For, since C and C_1 are the vertices of two equilateral triangles on AB, they are the centres of two circles which intersect in the points A, B, and have AB for a common chord. But (*Theor.* 6) the line CC_1 joining the centres of two intersecting circles bisects that chord. Therefore the line AB is bisected at M, *as required.*

7. *To erect a perpendicular to a directive X at any point O on it.*

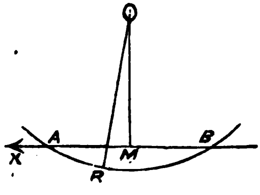
Take on X any two equal distances OA, OB, from O on opposite sides of O; and on AB, on its two opposite sides, construct two equilateral triangles, ACB and AC_1B. Join their vertices C, C_1. The joining line CC_1 passes through O, and is perpendicular to AB or to the directive X.



For, as proved in the last problem, the line CC_1 bisects AB, and therefore must pass through O, OA being by construction equal to OB. But CC_1, being also the line joining the centres of two circles of which AB is a common chord, is (*Theor.* 6) perpendicular to that chord, and CC_1 is therefore the perpendicular *required.*

8. *From a point O to draw a perpendicular to a given directive.*

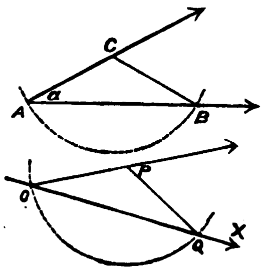
From O as centre and any distance OR terminating on the opposite side of the directive, describe a circle cutting the directive in the points A and B. Bisect (*Prob.* 6) the line AB at M, and join O with M. The line OM is the perpendicular required.



For, since AB is a chord of the circle ARB, and the line OM is drawn from the centre O to its bisection M, OM is (*Theor.* 4) perpendicular to AB, and therefore to the directive X, *as required.*

9. *To draw through a given point O on a given directive X a line making with it an angle equal to a given angle α.*

Let the angle α be acute; for, if obtuse, the supplementary acute angle may be substituted and the obtuse angle thereby determined.

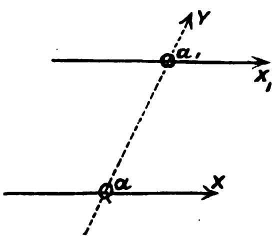


On one of the legs of the angle α take any distance AC. With C as centre and CA as radius, describe a circle which, α being acute, must cut AB in another point B. On the directive X take OQ equal to AB, and with O and Q as centres describe two circles with radii OP and QP, equal to AC and BC, intersecting at P. Join P with O and Q. The line OP makes the required angle α with X.

For, describing with P as centre, and either of the equal lines PO or PQ as radius, an arc of a circle passing through O and Q, the two circles, with C and P as centres, and CA and PO as radii, by construction, have their radii equal, and the chords AB and OQ also equal. The arcs on these chords are therefore equal (*Theor.* 11, *converse*), and the angles ACB and OPQ equal; and consequently the angles the chords make with the radii CA and PO (*Theor.* 2) are equal, that is, the angle POQ is equal to α, *as required.*

10. *To draw through a given point O a parallel to a given directive X.*

Through the point O draw any directive Y cutting X at any point Q at an angle α. Draw then, by the last problem, through O a directive X_1, making with Y an angle α_1 equal to the angle α between X and Y at Q. Then X_1 is the required parallel.

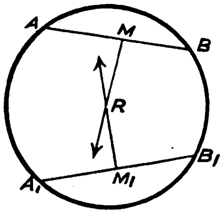


For since, by construction, α_1 is made equal to α, the directives X_1 and X are parallel (*Theor.* 10, *Chap. I.*), *as required.*

11. *To find the centre of a given circle R.*

Draw any two chords, AB, A_1B_1, of the given circle. Bisect

these chords (*Prob.* 6) at M and M_1. At M and M_1 then erect perpendiculars (*Prob.* 7) to these chords. The point R where these perpendiculars meet is the *required* centre.

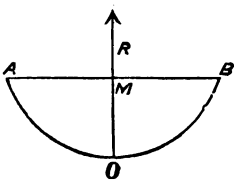


For since AB is bisected at M, and RM is perpendicular to it (*Theor.* 4), the centre is somewhere on the directive RM. Also, since A_1B_1 is bisected at M_1 and RM_1 is perpendicular to it, the centre must also be on RM_1. The centre is therefore on both RM and RM_1; and, being on both, it can be only at their intersection R.

COR. 1.—When only a portion of a circle, or an arc, is given, the centre is found by the same construction, the chords AB and A_1B_1 being taken within the limits of the arc; or by joining any point on the arc with its two extremities.

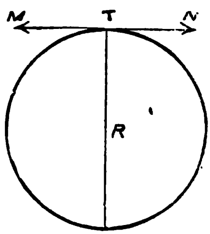
12. *To bisect an arc AB.*

Join the extremities of the arc by its chord AB. Bisect this chord at M, and erect at M a perpendicular MR to the chord. This perpendicular produced to meet the arc bisects it at O.



For the perpendicular OMR to a chord AB at its bisection passes through the centre R of the circle. And (*Theor.* 2) this perpendicular also bisects the arc, *as required.*

13. *At a point T on the circumference of a circle R to draw a tangent to the circle.*

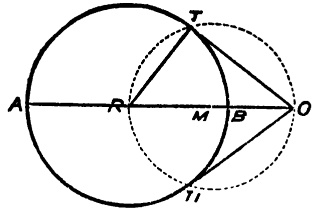


Join the centre R with the point T; and at T erect (*Prob.* 7) a perpendicular MN to the radius RT. The perpendicular thus erected is the tangent at T.

For the tangent at any point of a circle is (*Theor.* 15) perpendicular to the radius at that point.

14. *From a point O outside a given circle R to draw tangents to it.*

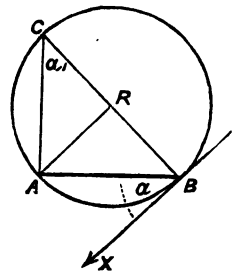
Join the centre R of the circle with O. Bisect RO at M, and, with M as centre and MR as radius, describe a circle cutting the circle R in T and T_1. Then join OT and OT_1. These lines OT, OT_1 are the tangents.



For, since RO is the diameter of the circle described with radius MR, and that circle passes through R, the angles RTO and RT_1O, being angles in semicircles, are (*Theor.* 12) each a right angle. Therefore the radii RT and RT_1 are at right angles to OT and OT_1 respectively. Consequently OT and OT_1 are (*Theor.* 15) the tangents from O, *as required.*

15. *On a given line AB to describe a segment of a circle which shall contain an angle* α *of given magnitude.*

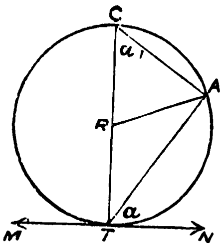
At the extremity B of AB draw a directive X (*Prob.* 9), making, with AB, an angle α equal to the given magnitude. Erect then a perpendicular BC at B to X, and also a perpendicular AC at A to AB, both meeting at C. Bisect BC at R; and, with R as centre and RB as radius, describe a circle. This circle will pass through A and C, and the segment ACB will contain an angle α_1 equal to α.



For, first, the angle BAC is by construction a right angle, and must (*Theor.* 12) have its vertex A on the semicircle CAB; secondly, the angle α_1 in the segment ACB is (*Theor.* 16) equal to that between X and AB, or to the given angle α, *as required.*

16. *To cut off from a given circle R a segment which shall contain an angle* α *of a given magnitude.*

Draw MN, a tangent to the circle R at any point T; and then draw TA (*Prob.* 9), making, with the tangent MN at T, an angle equal to the given magnitude α. The segment ACT contains an angle α_1 of the given magnitude.



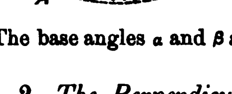
For, since MN is a tangent and TA a chord, the angle α is (*Theor.* 16) equal to α_1 in the alternate segment ACT; and the segment is cut off, *as required.*

CHAPTER III.

THE TRIANGLE AND PARALLELOGRAM.—THEOREMS.

THE principles established in the two preceding Chapters may now be applied to triangles and parallelograms.

1. *The Angles at the base of an Isosceles Triangle opposite the Equal Sides are equal.*



Let AB be the base, and AC, BC, the equal sides of the isosceles triangle ACB; and let also the dotted curve AB be the arc of a circle through A and B, having C for centre and CA or CB for radius.

Then, since the line AB is a chord of the circle, it makes (*Theor.* 2, *Chap II.*) equal angles with the radii CA and CB, through its extremities. The base angles α and β are therefore equal, *as stated.*

2. *The Perpendicular from the Vertex of an Isosceles Triangle on the Base bisects the base, and also the vertical angle.*

Using the same figure as before, let CM be the perpendicular from C on AB. Then (*Theor.* 2, *Chap. II.*) this perpendicular from the centre C of the circle AB on the chord AB bisects the angle ACB subtended by the chord, and bisects also the chord, that is, the perpendicular bisects both the base and the vertical angle of the isosceles triangle, *as stated.*

CONVERSELY, it follows that:—

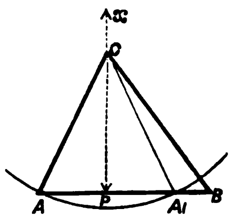
1. *The bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to it.*

2. *The bisector of the base of an isosceles triangle is perpendicular to the base, and also bisects the vertical angle.*

These are evident; for there can be only one bisector of the base, only one bisector of the vertical angle, and only one perpendicular on the base in any triangle. But in an isosceles triangle all these three coincide, as has been proved above.

3. *If the Perpendicular from the Vertex of any Triangle on its Base bisects the base, the triangle is isosceles.*

Let ACB be the triangle, and CP the perpendicular from the vertex C on AB. Then, if ACB be not isosceles, let CB be the greater side, and, with C as centre and CA as radius, let a circle be described cutting AB in A_1. Then the triangle ACA_1 is isosceles, and its base AA_1 (*Theor.* 2) is bisected at P. But, by supposition, AB is also bisected at P, and therefore both BP and A_1P are equal to AP. They are therefore equal to each other, which is impossible (*Ax.* 10). The triangle ACB is therefore isosceles.



CONVERSELY, it follows that:—

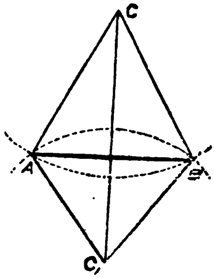
1. *All triangles on a base AB with their vertices on a perpendicular PCX at its bisection are isosceles.*

2. *Also, if a triangle stands on the same base with an isosceles triangle, and has its vertex on the bisector of the vertical angle of the isosceles triangle, both triangles are isosceles.*

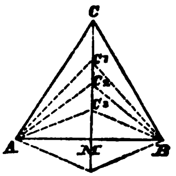
For the bisector of the vertical angle of the given isosceles triangle is also the perpendicular on the base and the bisector of the base.

4. *If Two Isosceles Triangles stand on the Same Base, the Line Joining their Vertices bisects the common base, and is perpendicular to it; and also bisects both vertical angles.*

Let AB be the common base, and ACB, AC_1B the two isosceles triangles. Then, supposing circles described through A, having C and C_1 for centres, and CA and C_1A for radii, these circles pass also through B, and AB is a common chord of them; but (*Theor.* 6, *Chap. II.*) the line CC_1, joining the centres of two intersecting circles, bisects the common chord AB, and bisects also the angles C, C_1 at the centres subtended by this chord. The line CC_1, therefore, joining the vertices of the two triangles, bisects their common base and is perpendicular to it, and also bisects the two vertical angles, *as stated.*



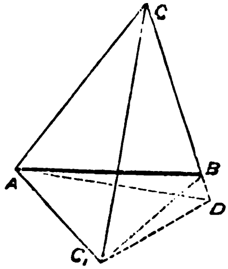
COR. 1. Hence, if any number of isosceles triangles stand on the same base AB, their vertices C, C_1, C_2, &c., are on a directive MC perpendicular to the base at its bisection M, as in the Figure.



COR. 2. Hence, also, the line CM bisects all the angles C, C_1, C_2, &c., and the angles also of any other isosceles triangles constructed either above or below AB.

5. *If Two Triangles stand on a Common Base, and the Line Joining their Vertices bisects both vertical angles, the triangles are isosceles.*

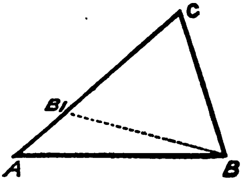
Let AB be the common base, and ACB, AC_1B the two triangles, and CC_1 the line bisecting both vertical angles C and C_1. Then, if possible, suppose the triangle ACB not to be isosceles, and AC greater than BC. Take on CB produced a point D, such that CD be equal to CA. Then the triangle ACD is isosceles; and, since its vertical angle ACD is bisected, the line CC_1 is (*Theor.* 2, *converse*) perpendicular to AD. But the triangle AC_1D is also isosceles (*Theor.* 3, *converse*), being on the base AD, and having its vertex on the perpendicular CC_1 to AD.



Therefore its vertical angle AC_1D is bisected by CC_1; and the angle DC_1C is equal to AC_1C. But, by supposition, the angle BC_1C is equal to AC_1C; therefore BC_1C should be equal to DC_1C, a part equal to the whole (*Ax.* 10). The two triangles therefore are isosceles, *as stated.*

6. *If, in any triangle, One Side be Greater than another, the Angle Opposite the Greater Side is greater than the angle opposite the less side.*

Let ABC be the triangle, the side AC greater than BC; and let B_1C be equal to BC. Then the triangle BB_1C is isosceles. But BB_1C being isosceles, the angles BB_1C and B_1BC (*Theor.* 1) are equal. But the angle ABC is greater than B_1BC, and therefore greater than BB_1C, which is equal to B_1BC. And also the angle BB_1C is (*Theor.* 12, *Chap. I.*) greater than the internal angle B_1AB. Therefore the angle ABC, which is greater than BB_1C, is greater than the angle BAC, *as stated.*



It thus having been proved (*Theor.* 1) that, when the sides of a triangle are equal, the opposite angles are equal; and (*Theor.* 6) that, when the sides are unequal, the greater angle is opposite the greater side, it follows, by the principle of Double Conversion (*Introduction*) that:—

1. *When Two Angles of a Triangle are Equal, the sides opposite these angles are equal.*

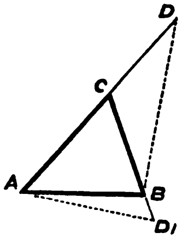
2. *When One Angle of a Triangle is Greater than Another, the side opposite the greater angle is greater than the side opposite the less.*

COR. 1. Hence, in a right angled triangle, the hypothenuse which is opposite the greatest angle is greater than either of the other sides.

COR. 2. Hence, also, the shortest line which can be drawn from a point to a directive is the perpendicular from the point on the directive; for any other line is the hypothenuse of a right angled triangle, of which the perpendicular is a side.

7. *The Sum of Two Sides of a Triangle is greater than the third side, and their Difference less.*

FIRST PART.—Let ACB be the triangle, and on AC produced let CD be equal to CB. Then, since CD and CB are equal, the angles CDB and CBD (*Theor.* 1) are equal. But the angle ABD is greater than CBD by construction, and therefore greater than CDB. Therefore (*Theor.* 6), in the triangle ABD, AD, opposite the greater angle ABD, is greater than AB opposite the less angle ADB. But AD is the sum of the sides, which therefore is greater than AB, the base, *as stated.*



SECOND PART.—In the same figure let CB be produced to D_1, so that CD_1 be equal to the greater side AC. Then BD_1 is the difference of the sides AC and BC. But the triangle AD_1C, being isosceles, the angle AD_1C is equal to D_1AC, and is therefore greater than the angle BAD_1. Therefore the side BD_1, the difference, opposite the less angle BAD_1, is less than the base AB opposite the greater angle AD_1B. Hence, the difference of the sides is less than the base.

ANOTHER PROOF OF THE SECOND PART.—Since in *the First Part* the sum of two sides is greater than the third, if we take one of the two sides in the sum from both magnitudes, the one side remaining from the sum is greater than the difference of the third side and that taken from it.

TWO TRIANGLES.

In order that two triangles should be equal, three conditions are invariably necessary ; but these must be *independent* conditions, that is, no one of the given conditions can be a consequence of the other two. Thus, to be given the three angles of two triangles equal, is not to be given three independent conditions ; for, when two of them are known, the third is necessarily known from the sum of the three angles of a triangle being equal to two right angles.

The cases in which two triangles are invariably equal are :—

1st. When two triangles have their three pairs of sides equal respectively.

2nd. When two triangles have two pairs of sides equal, and the angles contained by them equal.

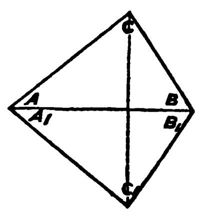
3rd. When two triangles have one pair of sides equal, and any two pairs of angles respectively equal, which comes to the three angles being equal.

4th. When two triangles have two pairs of sides equal, and the angles opposite one pair of sides also equal, and the angles opposite the other pair of equal sides are either both acute or both obtuse.

It should be observed that, in the third case, two pairs of angles being, by supposition, equal, all three pairs of angles must be equal; and therefore the third case can be proved always as that of one pair of equal sides, and two pairs of angles *adjacent* to these sides also equal.

The fourth case has a condition annexed; that the angles opposite the remaining pair of sides should be both obtuse or both acute. On this account it is known as the "Ambiguous Case," there being, when the angles opposite the remaining pair of sides are both acute or both obtuse, an equality of the triangles; but when they are one acute and the other obtuse, an inequality.

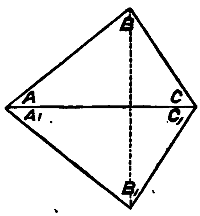
8. *If Two Triangles have their Three pairs of Sides respectively Equal, the angles opposite the equal sides are equal.*



Let ABC and $A_1B_1C_1$ be the two triangles, placed so that a pair of sides, AB, A_1B_1, may coincide and the triangles be on opposite sides of AB. Then, since AC is equal to A_1C_1, the triangle CAC_1 is isosceles. Also, since BC is equal to B_1C_1, the triangle CBC_1 is isosceles. But (*Theor.* 4) the line AB, joining the vertices of these two isosceles triangles, bisects their vertical angles CAC_1 and CBC_1. Therefore, the angles A and A_1 are equal, and B and B_1 also equal. But these are two pairs of the angles of the triangles; and the third pair are necessarily equal, *as stated.*

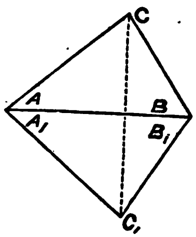
9. *If Two Triangles have Two pairs of Sides in each respectively Equal, and the Angles Contained by these sides also equal, the remaining sides and angles are equal.*

Let ABC and $A_1B_1C_1$ be the two triangles, placed so that one pair of equal sides, AC and A_1C_1, may coincide. Then, since BC is equal to B_1C_1, the triangle BCB_1 is isosceles; and, the angle C being equal to C_1, the line AC is the bisector of the vertical angle BCB_1 of the isosceles triangle BCB_1. Therefore (*Theor.* 3, *converse*) the triangle BAB_1 with its vertex on this bisector, and BB_1 for base, is isosceles; and the angle A is equal to angle A_1, and the sides AB and A_1B_1 are equal, *as stated.*



10. *If Two Triangles have One pair of Sides Equal, and the Two Angles Adjacent to the equal sides also respectively equal, the remaining sides and angles are equal.*

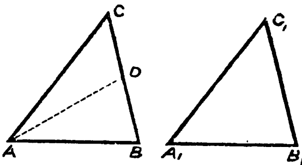
Let ABC and $A_1B_1C_1$ be the two triangles so placed that the equal sides AB and A_1B_1 may coincide, and A, A_1 and B, B_1 be the pairs of equal angles. Then, since CAC_1 and CBC_1 are two triangles on opposite sides of a common base CC_1, and the line AB joining their vertices divides the angles CAC_1 and CBC_1 into the equal angles A, A_1 and B, B_1, the two triangles CAC_1 and CBC_1 (*Theor.* 5) are isosceles. And therefore the sides AC and A_1C_1 are equal; and BC and B_1C_1 are also equal. The angles C and C_1 are necessarily equal, since the angles A and B are equal to A_1 and B_1.



11. *If Two Triangles have Two pairs of Sides Equal, and the Angles opposite One pair of Sides also equal, and the angles opposite the other pair of equal sides either both Acute or both Obtuse, the remaining sides and angles are equal.*

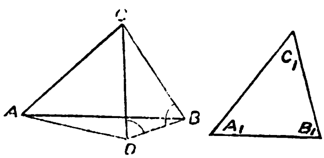
Let ABC and $A_1B_1C_1$ be the two triangles in which AC and AB are equal to A_1C_1 and A_1B_1, and the equal angles C and C_1

are opposite to AB and A_1B_1. Then, suppose, if possible, that the angle A is greater than A_1; and let AD be a line making, with AC, an angle, CAD, equal to A_1. Then; since in the triangles CAD and $C_1A_1B_1$ the angles C and CAD are equal to C_1 and A_1, and the sides AC and A_1C_1 are equal (*Theor.* 10), the side AD is equal to A_1B_1, and the angle ADC to B_1. Also, since AD is equal to A_1B_1, it is equal to AB; and AD and AB are the equal sides of an isosceles triangle ADB. The angles ADB and ABD are therefore equal, and both acute. Therefore the angle ADC is obtuse, and consequently B_1 is obtuse; which is impossible, since B is acute, and the theorem requires that B and B_1 be either both acute or both obtuse. The remaining sides and angles of the triangles are therefore equal, *as stated.*



12. *If Two Triangles have Two pairs of Sides respectively Equal; and if the Angle Contained by one pair of sides in one triangle be Greater than the angle contained by the other pair of sides in the other, then the side opposite the greater angle is greater than the side opposite the less.*

Let ABC and $A_1B_1C_1$ be the two triangles; the sides AC and BC respectively equal to A_1C_1 and B_1C_1, but the angle ACB greater than the angle $A_1C_1B_1$. Let, also, CD be a line equal to CB or C_1B_1, making with AC an angle equal to $A_1C_1B_1$. Then, since in the two triangles ACD and $A_1B_1C_1$ the sides AC and CD are respectively equal to A_1C_1 and B_1C_1, and the angle ACD equal to $A_1C_1B_1$, the third pair of sides, AD and A_1B_1, are (*Theor.* 9) equal. But in the triangle ADB, the angle ADB is greater than CDB; also the angle CDB is equal to angle CBD, and is greater than angle ABD. Therefore, the angle ADB is much greater than angle ABD; and (*Theor.* 6, *Converse*) the side AB is greater than the side AD, that is, greater than the side A_1B_1, *as stated.*



Hence by the principle of Double Conversion

(Introduction) it may be inferred that, since, when two triangles have two pairs of sides and the angles between them equal, the third pair of sides are equal, and when two triangles have two pairs of sides equal and the angle between the sides in one triangle greater than the angle between the sides in the other, the side opposite the greater angle is greater than the side opposite the less, so—

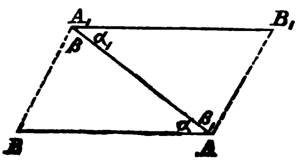
When Two Triangles have Two pairs of Sides respectively Equal, and the Third Side in one is Greater than the third side in the other, the Angle Opposite the greater side is greater than the angle opposite the less.

The next subject is that of the Parallelogram, which is a case of two equal triangles reversed on opposite sides of a common base, termed "the Diagonal." Its peculiar properties depend on the theorems established as to two triangles.

PARALLELOGRAMS.

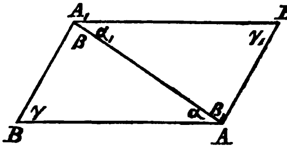
13. *Lines which Join the extremities of Two Equal and Parallel lines are themselves equal and parallel.*

Let AB, A_1B_1 be the equal and parallel lines, and let their extremities be joined by the lines AB_1 and A_1B; and suppose AA_1 joined. Then in the two triangles AA_1B and AA_1B_1 the two sides AB and A_1B_1 are by supposition equal, and the side AA_1 is common. Also, since AB is parallel to A_1B_1, the angles α and α_1 are equal. The two triangles therefore, AA_1B and AA_1B_1 (*Theor.* 9), are equal; and the sides AB_1 and A_1B are equal; and also the alternate angles β and β_1 are equal, that is, AB_1 and A_1B are equal and parallel, *as stated.*



14. *The Opposite Sides of a Parallelogram are Equal to each other, also the Opposite Angles.*

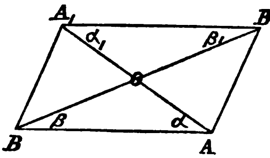
Let ABA_1B_1 be the parallelogram, and AA_1 one of its diagonals. Then, since AB and A_1B_1 are parallel (*Def. of Prlgm.*), the angles α and α_1 are equal. Also, since AB_1 and BA_1 are parallel, the angles β and β_1 are equal. Therefore in the two triangles, ABA_1 and AB_1A_1, the side AA_1 is common and the angles adjacent to AA_1 equal respectively. Therefore, the two triangles ABA_1 and A_1B_1A are (*Theor.* 10) equal;



and the side AB equal to A_1B_1, and side AB_1 equal to A_1B. The opposite angles γ, γ_1 are also equal, and the angles at A and A_1, composed of α, β_1, and α_1, β, are equal.

15. *The Diagonals of a Parallelogram Bisect each other.*

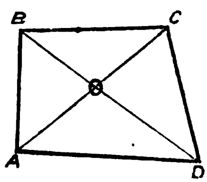
Let ABA_1B_1 be the parallelogram, and AA_1, BB_1 the diagonals meeting at O. Then, in the two triangles AOB and A_1OB_1, the sides AB and A_1B_1 are (*Theor.* 14) equal, and the angles α and β are equal respectively to the angles α_1 and β_1. Therefore (*Theor.* 10), the remaining sides are equal, AO to A_1O and



BO to B_1O; that is, the diagonals are bisected, *as stated.*

16. CONVERSE.—*If the Diagonals of a Quadrilateral Bisect each other, the quadrilateral is a parallelogram.*

Let $ABCD$ be the quadrilateral, and AC, BD the diagonals bisecting each other at O. Then, since AO is equal to CO and BO to DO, in the two triangles AOD and BOC, the sides AO and DO are equal respectively to CO and BO, and the contained angles AOD and COB also equal. The other sides and angles are (*Theor.* 9) therefore equal; BC equal to AD and the angle OCB equal to the angle OAD. The sides AD and BC

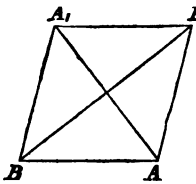


are therefore equal and parallel, and the other sides AB and CD

are (*Theor.* 13) also equal and parallel, and the quadrilateral is therefore a parallelogram, *as stated.*

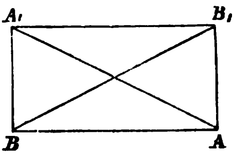
17. *If the Opposite Sides of a Quadrilateral be Equal, it is a parallelogram.*

Let ABA_1B_1 be the quadrilateral. Then, since AB is equal to A_1B_1, and AB_1 to A_1B and AA_1 a common side in the two triangles ABA_1 and A_1B_1A, the angles of these triangles (*Theor.* 8) are equal; the angle BAA_1 equal to B_1A_1A, which makes AB parallel to A_1B_1; and the angle AA_1B equal to A_1AB_1, which makes AB_1 parallel to A_1B. The quadrilateral is therefore a parallelogram, *as stated.*



18. *The Diagonals of a Rectangle are Equal.*

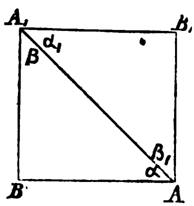
Let the rectangle be ABA_1B_1, and the diagonals AA_1 and BB_1. Then, in the two triangles ABB_1 and ABA_1, the side AB is common, and the sides AB_1 and A_1B are equal; and the angles contained by these sides are equal, being right angles. Therefore the third pair of sides (*Theor.* 9), AA_1 and BB_1, are equal; that is, the diagonals are equal, *as stated.*



COR.—As squares are rectangles, it follows that their diagonals are equal.

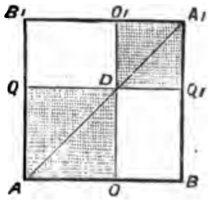
19. *The Diagonals of a Square make each Half a Right Angle with the sides of the square.*

Let ABA_1B_1 be the square, and AA_1 one of its diagonals. Then, in the triangle ABA_1, since the two sides AB and A_1B are equal, the triangle ABA_1 is isosceles. But the angle at B is a right angle. Therefore (*Theor.* 14, *Cor.* 7, *Chap. I.*) each of the base angles α, β, is half a right angle. In like manner, in the triangle AB_1A_1, the angles β_1 and α_1 are each half a right angle; and the four angles α, β, α_1, β_1, are each half a right angle, *as stated.*



20. *The Rectangles about the Diagonals of a Square are Squares; and the Complementary Rectangles have equal sides, equal to the sides of the squares.*

Let ABA_1B_1 be the square, and OO_1, QQ_1 lines drawn through a point D on the diagonal AA_1 parallel to the sides AB_1 and AB of the square. Then (*Theor.* 19) the angles BAA_1 and B_1A_1A are each half a right angle, and also angles B_1AA_1 and BA_1A each half a right angle. Moreover, angles QDA and ODA are (*Theor.* 10, *Cor.* 1, *Chap. I.*) equal respectively to OAD and QAD, and therefore each half a right angle. Therefore the sides QD and QA are equal, also OD and OA. Also QD (*Theor.* 14) is equal to OA, and OD to QA. The four sides of the figure $AODQ$ are therefore all equal, and its angles all right angles; and the figure is a square. Similarly, it is proved that the four sides of $A_1O_1DQ_1$ are equal and the figure a square.



Also, in the rectangles BD and B_1D, the sides OB and Q_1B are equal to the sides DQ_1 and DO of the squares A_1D and AD, and the sides B_1Q and O_1B_1 equal to the sides DO_1 and DQ of the same squares. The complementary rectangles are therefore equal to two rectangles under the sides of the squares.

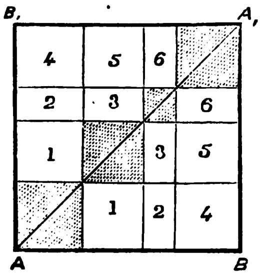
COR.—Hence, it is evident that the four sides of the square AA_1 are cut all round into pairs of segments, those of each pair equal respectively to AO and BO.

The principle may be extended thus :—

21. *A Square being Divided into any number of Rectangles about the Diagonal and pairs of Complementary Rectangles, the rectangles about the diagonals are squares, and the pairs of complementary rectangles have their sides equal to corresponding pairs of sides of the squares.*

Let ABA_1B_1 be the square, and suppose it divided by parallels through points along the diagonal AA_1 into rectangles (shaded in the Figure) about the diagonal and their pairs of complementary rectangles, marked 1, 2, 3, 4, 5, 6. Then, by the same proof as above, it can be shown that, in all the rectangles

about the diagonal, the diagonal of each makes half a right angle with its sides, and that each is therefore a square.

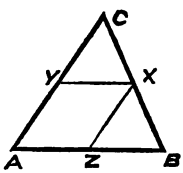


The rectangles also, as numbered, have their sides equal to the sides of a pair of squares. Those of rectangles No. 1 being the sides of the 1st and 2nd squares counted from A; of No. 2, sides equal to the sides of the 1st and 3rd squares; those of No. 3, the sides of the 2nd and 3rd squares, and so on; observing that the sides of any of these rectangles may be ascertained by looking to the squares to right or left and upwards or downwards. For example, rectangles No. 5 lead upwards and downwards to the 2nd and 4th squares and sideways to the very same squares.

COR.—Hence, it follows that the sides AB, A_1B_1, BA_1 and AB_1 are all divided similarly; and that the segments of any one are equal respectively to the segments of the other, all round.

22. *The Line from the Bisection of a Side of a Triangle Parallel to the Base bisects the other side, and is half the base.*

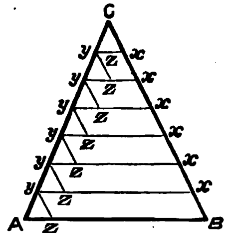
Let ACB be the triangle, AB the base, X the bisection of the side BC, and Y the point in which the parallel to AB from X meets AC. Let also XZ be a parallel to AC. Then, in the two triangles XYC and XZB, the sides XC and XB are equal; and, since XY and XZ are parallels to AB and AC, their angles also are equal respectively. Therefore (*Theor.* 10) their remaining sides are equal, XY to BZ and YC to XZ. But the figure $AZXY$ is a parallelogram (by construction); therefore AZ is equal to YX, which is proved equal to ZB. The base AB is therefore bisected at Z, and XY is half AB. Also AY is equal to ZX, which is equal to YC; and therefore AC is bisected at Y, *as stated.*



It is also evident that, if Y were joined with Z, the whole triangle ABC would be divided into four triangles, all having their sides respectively equal.

23. *If a Side of a Triangle be Divided into any number of Equal Parts, and from the points of section Parallels be drawn to the Base, these parallels divide the other side into the same number of equal parts.*

Let ACB be the triangle, AB the base, and let the side BC be divided at the intervals marked x into equal parts, and the lines xy be parallels to AB, dividing AC into segments at the points marked y; and yz lines parallel to BC, forming a series of triangles yzy equal to the number of parts into which BC is divided. Then, since the lines yz are parallel to BC, and the lines xy to AB, the figures $xyzx$ are parallelograms, and the lines yz are equal to the lines xx. But the triangles yzy are all equiangular, their sides being parallel to those of ACB. Their sides yz are also equal, as now proved. Therefore their remaining sides (*Theor.* 10) are equal, and the lines yy consequently equal; that is, AC is divided into equal parts equal in number to the parts of BC, *as stated.*

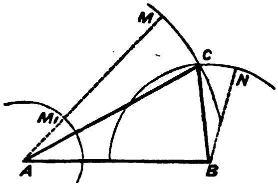


Cor.—It is evident that, if the lines yz were produced to meet AB, they would divide that line into the same number of equal parts.

PROBLEMS.

1. *The three sides of a triangle being given in magnitude, to construct the triangle.*

Let AB be one of the sides of given magnitude. Set off (*Prob.* 2, *Chap. II.*) two lines AM, BN from A and B equal to the two other given magnitudes; and with A and B as centres, and AM and BN as radii, describe circles. If these circles intersect in a point C, then ACB is the required triangle.

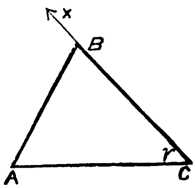


For, since AM and BM are, by construction, equal to two of the sides of given magnitude, and AB is the third given side, the triangle has its three sides of the given magnitudes.

N.B.—If the two circles do not intersect, as illustrated by the smaller circle with AM_1 as radius, the problem is impossible; the sum of the two given sides AM_1 and BN being less than the base AB, contrary to *Theor.* 7.

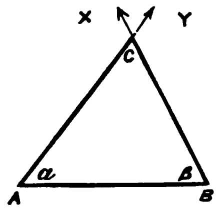
2. *Two sides of a triangle being given in magnitude and also the angle contained by these sides, to construct the triangle.*

Let AC be one of the sides of given magnitude; and through C draw (*Prob.* 9, *Chap. II.*) a directive X, making with AC an angle γ of the given magnitude. Take on X a distance CB equal to the other given side, and join BA. The triangle ACB is the one required.



For, the sides AC and BC, and the angle γ, being constructed of the given magnitudes, no other triangle (*Theor.* 9) can be made of them of different remaining side or angles.

3. *One side of a triangle being given in magnitude and also the two angles adjacent to that side, to construct the triangle.*

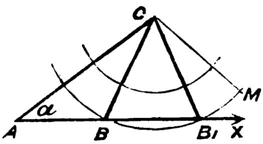


Let AB be the side of the given magnitude; through A and B (*Prob.* 9, *Chap. II.*) draw directives Y and X, making with AB angles α and β of the given angular magnitudes. Let these directives meet in C. Then the triangle ACB is the one required.

For the side AB and the angles α and β being those given in magnitude, no other triangle (*Theor.* 10) can be constructed having different remaining sides or angle.

N.B.—If the side AB and two angles, not both adjacent to AB, were given in magnitude, the solution would be the same; for the third angle, which must then be adjacent to AB, is necessarily known in magnitude, the three angles of a triangle being two right angles; and the problem would still be that of a side and two adjacent angles.

4. *Two sides of a triangle being given in magnitude and the angle opposite one of the sides, to construct the triangle.*



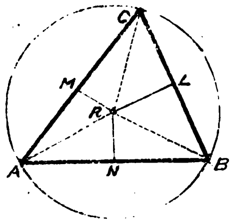
Let AC be the side adjacent to the given angle. From its extremity A draw a directive X, making with AC an angle α equal to the given angle. Then, with C as centre and a line CM set off, (*Prob.* 2, *Chap. II.*) equal to the other given side, as radius, describe a circle cutting X in B and B_1. Either of the triangles ABC or AB_1C is the one required.

For these two triangles both satisfy the conditions of the problem, having the given sides and angle. There are thus two solutions, for which reason this is known as the "ambiguous case" (*see Theor.* 11).

N.B.—The problem is impossible if the side CB be less than the perpendicular that may be dropped from C on X, as illustrated in the diagram, by the smaller circle round C. If CB be equal to that perpendicular, there is only one solution, a right-angled triangle, when B_1 coincides with B.

5. *To describe a circle through the vertices A, B, C of a given triangle.*

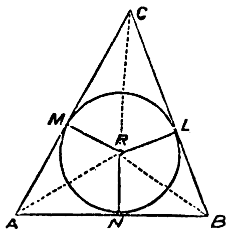
Bisect the sides BC and AC in L and M, and at these points of bisection erect perpendiculars LR and MR to BC and AC to meet in R. With R, then, as centre, and RC as radius, describe a circle. This circle will pass through the vertices A, B and C, as required.



For, since LR is perpendicular to BC at its bisection L, the triangle BRC is (*Theor.* 3, *converse*) isosceles, and therefore BR is equal to CR. So likewise, MR being perpendicular to AC at its bisection M, the triangle ARC is isosceles and AR equal to CR. Hence AR, BR, and CR are equal to each other; and the circle with R as centre and RC as radius passes through the vertices A, B, and C.

6. *To inscribe a circle in a triangle ABC touching its sides.*

Bisect the angles A and B, and let the bisectors AR and BR meet at R. Drop from R a perpendicular RN on AB. With R as centre and RN as radius describe a circle. This circle will touch the three sides of the triangle and be the one required.

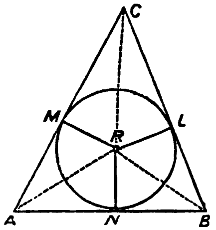


For, drawing RL and RM perpendicular to BC and AC, in the two triangles RMA and RNA, the angles at M and N are equal, being right angles, and the side AR is common, and the angles RAM and RAN are equal. Therefore the sides RM and RN are equal. In like manner, since the angle B is bisected by BR, the perpendicular RL is equal to RN. The circle, with R as centre and RN as radius, therefore passes through L, M, and N; and, its radii RL, RM, and RN being perpendicular to BC, CA, and AB at these points, the circle touches the sides of the triangle.

7. *To circumscribe a triangle to a given circle R, its angles being of given magnitudes.*

From the centre R of the circle draw any radius RL, and then draw also two other radii RM and RN, making with RL angles equal to the supplements of two of the angles of given magnitude of the required triangle. At L, M, and N draw then tangents

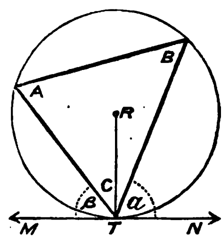
to the circle R. The triangle ABC formed by these tangents is the one required.



For the angles at L, M, and N, being right angles in the quadrilateral $MRLC$, the sum of the angles MCL and MRL is two right angles; and MCL is the supplement of MRL, and therefore (*by construction*) one of the angles of the required triangle. In like manner is NBL the supplement of NRL, and another angle of the required triangle.

8. *To inscribe in a circle R a triangle, two angles of which are given in magnitude.*

At any point T on the circle draw a tangent MN; and then draw (*Prob.* 9, *Chap. II.*) two lines, TA and TB, chords of the circle, making with MN angles β and α equal to those of the given magnitudes. Join A with B. The triangle ACB is the one required.

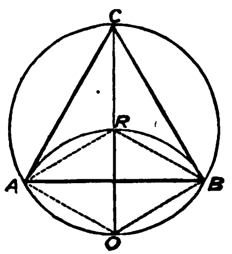


For, since the chords TA and TB make (*by construction*) angles β and α, as given, with MN, the angles A and B in the alternate segments are of the given magnitudes, and the triangle is as required.

The inscribing of an equilateral triangle in a circle may be effected in the same way as above, by making α and β each two-thirds of a right angle; but a special construction for it seems desirable, as follows :—

9. *To inscribe an equilateral triangle in a given circle R.*

With any point O on the circle R as centre, and OR as radius, describe a circle, cutting the given circle in A and B. Join A with B, and produce OR to meet the given circle in C. Then join C with A and B. The triangle ACB is the equilateral required.

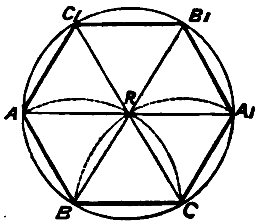


For, since OA and OB are equal, by construction, to RA, RO, and RB, the triangles ARO and BRO are equilateral. Therefore, the angle ARB is four-thirds of a right angle; and the angle ACB at the circumference is (*Theor.* 12,

Chap. II.) two-thirds of a right angle. Also, since ARB and AOB are isosceles triangles standing on the base AB, and the triangle ACB has its vertex on ORC the line joining their vertices, it is (*Theor.* 4, *Cor.* 1) also isosceles, and its base angles each two-thirds of a right angle. It is therefore equilateral.

10. *To inscribe a regular hexagon in a given circle R.*

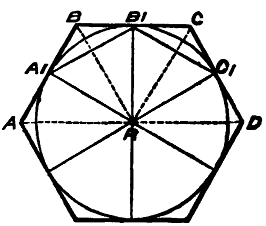
With any point B on the circle R, and BR as radius, describe a circle cutting the given circle in A and C. This circle passes also through the centre R. Again, with C as centre and CR as radius, describe another circle cutting the given circle in A_1. This circle passes through R and B. Join B and C with R and produce the joining lines to meet the circle in B_1 and C_1. The figure $ABCA_1B_1C_1$ is the required hexagon.



For, from the construction, the three triangles ARB, BRC, and CRA_1 are equilateral triangles, and the three chords, AB, BC, and CA_1, are equal—each equal to the radius. Further, the angles ARB, BRC, and CRA_1 are each two-thirds of a right angle, and the three together two right angles; and therefore the lines AR, RA_1 form one right line, a diameter. The vertically opposite angles A_1RB_1, B_1RC_1, and C_1RA are, consequently, each two-thirds of a right angle; and therefore the chords AC_1, C_1B_1, and B_1A_1 are equal, being equal each to AB or the radius. The angles also of the hexagon are also equal, being, evidently, each four-thirds of a right angle. The figure is therefore a *regular* hexagon.

11. *To circumscribe a regular hexagon to a circle R.*

Inscribe in the circle R (*last Prob.*) a regular hexagon, $A_1B_1C_1$, &c., and at its vertices draw tangents to the circle, meeting consecutively in the points A, B, C, D, &c., round the circle. The figure $ABCD$, &c., thus formed, is the required hexagon.



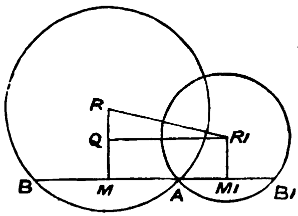
For, taking the quadrilateral A_1RB_1B, since BA_1 and BB_1 are tangents, from B to the circle, BR bisects the angle A_1BB_1. But, since the angles at A_1 and B_1 are right, the angle A_1BB_1 is the *supplement* of A_1RB_1, which, being

two-thirds of a right angle (*last Prob.*), the angle A_1BB_1 is four-thirds, and therefore RBB_1 and RBA_1 each two-thirds. In like manner, in the quadrilateral B_1RC_1C, the angle B_1CR is two-thirds of a right angle. The triangle BRC is therefore equilateral, and BC equal to BR. So, also, it may be shown that AB is equal to BR, and therefore equal to BC; and so round the figure its sides may be all proved equal, and its angles each four-thirds of a right angle. It is therefore a *regular* hexagon circumscribed.

The figure has not been lettered all round, as the principle of the proof is manifest from the upper half of the hexagon.

12. *To draw through an extremity A of the common chord of two intersecting circles R and R_1, a line of a given magnitude terminated by both circles.*

On the line RR_1 joining the centres construct a right-angled triangle RR_1Q, the side R_1Q of which is half the line of given magnitude. Through A then draw the line BB_1 parallel to R_1Q, and terminated on both circumferences. The line BB_1 is the one required.



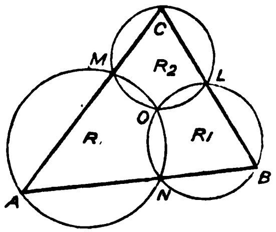
For; since BB_1 is parallel to R_1Q, the perpendicular RQ to R_1Q, produced to meet BB_1, is perpendicular to it. It therefore bisects at M the chord AB of the circle R. If from R_1 a perpendicular RM_1 be drawn to the chord AB_1, it will bisect the chord AB_1, and the line MM_1 be half the given magnitude. The double chord BB_1 is therefore equal to the given magnitude, as required.

COR. 1. When the line BB_1 is drawn parallel to RR_1, it is evident that it is the greatest possible that can be drawn. Hence, to draw the greatest possible double chord is to draw it parallel to the line joining the centres.

COR. 2. It is further evident that the least chord terminated by the two circles must be when that chord is perpendicular to RR_1; the side R_1Q being then cipher, and consequently BB_1 being cipher. This happens when BB_1 becomes the common chord of the two circles.

13. *To construct a triangle, its three sides being given in magnitude, so that they pass one through each of three given points, L, M, and N.*

The sides of the triangle being given, it can anywhere be constructed, and its angles are then known. Then, on the lines *LM* and *LN* describe segments (*Prob.* 15, *Chap. II.*) *LCM* and *LBN*, containing two of the ascertained angles *C* and *B*. Then draw (*last Prob.*) through *L* a line *BC* equal to the side opposite the remaining angle of the triangle. Join *C* with *M* and *B* with *N*, and produce the joining lines to meet at some point *A*. The triangle *BAC*, so formed, is the one required.



For; by the construction, the angles *C* and *B* are of the ascertained magnitudes, and *BC* is equal to one of the given sides. The triangle *ABC* is therefore equal to the given triangle and passes through *LM* and *N*.

N.B.—This problem may be put in another way, viz., being given the angles of a triangle and one side, to construct it so that its sides may pass one through each of the points *L*, *M* and *N*.

Hence the greatest triangle of given angles through *L*, *M* and *N* is that which has its three sides parallel to the lines joining the centres of the three circles R, R_1, R_2.

The student may also prove, as an exercise, that the circle *R* through *M* and *N*, and the intersection *O* of the circles R_1 and R_2, passes through the vertex *A* of the constructed triangle *ABC*.

EXERCISES.

1. Draw through a point on a circle a chord of a given magnitude.

2. Draw through a point, inside or outside a circle, a line, so that its segment intercepted by the circle be of a given magnitude.

3. Inscribe a square in a circle.

4. Exscribe a circle to a square.

5. Circumscribe a square to a circle.

6. Circumscribe a circle to a square.

7. Describe a square on a given line.

8. Draw two equal lines from two given points to a given directive or circle.

9. Draw from two given points lines to a given directive so that they make equal angles with it.

10. Prove that the sum of the two lines which make the equal angles with the directive is the least possible sum which can be drawn when the two points are on the same side of the directive; and that their difference is greatest when the two points are on opposite sides.

11. Prove that the sum of the perpendiculars dropped from two given points upon any directive is double the perpendicular upon the directive from the bisection of the line joining the two points.

12. Given base and vertical angle of a triangle, construct it so that its vertex be on a given directive or circle. When is the problem impossible?

13. The perpendiculars on two directives drawn from any point on the bisectors of either angle between them, are equal.

14. Hence prove that the three bisectors of the internal angles of a triangle meet in a point.

15. Also prove that a bisector of one internal angle of a triangle and the two bisectors of the external angles adjoining the other two internal angles, meet in a point.

16. Hence, find the centres of the four circles which may be described touching three intersecting directives.

17. The three perpendiculars from the vertices of a triangle on the opposite sides, meet in a point.

18. The point of meeting of these perpendiculars is the centre of the circle inscribed in the triangle formed by joining the feet of the perpendiculars.

19. Any two bisectors of two sides of a triangle drawn from the angles opposite these sides, trisect each other.

20. The three bisectors of the sides of a triangle drawn from the opposite angles meet in a point.

CHAPTER IV.

AREAS.*

The equality of areas is primarily established by a *supposed* superposition. When the boundaries of two closed figures, whether composed solely of lines or solely of curves, or partly of lines and partly of curves, may be made to coincide all round, the superficial spaces, or areas (*Lemma* 2), must be equal. Thus, two circles of equal radii having the same centre must coincide in their circumferences, and include equal areas. And even if they have different centres, their areas are equal; since, by superposing the centre of one on that of the other, the circumferences of both must coincide, and the two circles become practically one, and therefore have equal areas.

In like manner, two figures composed of two equal arcs of the same circle, and of two equilateral triangles constructed on the chords of these arcs, have equal areas.

By this kind of superposition Euclid proves (Bk. I. Prop. 4) that when two triangles have two pairs of sides respectively equal, and the angles contained by them equal, their areas are equal; and that also (Bk. I. Prop. 8) when two triangles have their three pairs of sides equal respectively, their areas are equal.

We take first the four fundamental cases of two equal triangles, considered in the Theorems 8, 9, and 10 of the preceding chapter, and prove that, in all, the areas of these triangles are equal. In all four cases

---

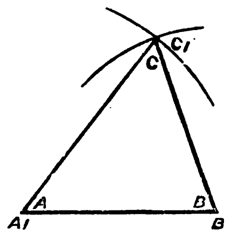
* *See* Appendix, Note 3.

(the fourth of which is reduced to the third) it is shown that, whatever be the three pairs of magnitudes (side or angle) given equal, the result is that all sides and angles are equal respectively. It is sufficient, therefore, to prove that, when the three pairs of sides of two triangles are equal respectively, the areas of the triangles are equal.

In the fifth or ambiguous case (*Theor.* 11, *Chap. III.*), the equality of *all* the sides and angles being subject to a condition, the equality of the areas cannot with certainty be affirmed.

1. *If Two Triangles have the Three Sides in one equal respectively to the three sides in the other, their areas are equal.*

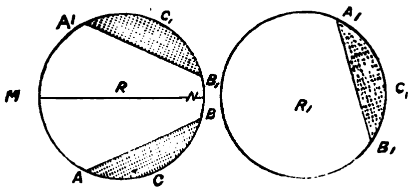
Let the two triangles ABC and $A_1B_1C_1$ be supposed so placed that their equal bases AB and A_1B_1 may coincide throughout, and their vertices C and C_1 may be on the same side of AB; and, with A as centre and AC as radius, suppose a circle described. Then, since A_1C_1 is equal to AC, the extremity C_1 of A_1C_1 will be on that circle. In like manner, suppose a circle described with B as centre, and BC as radius, the extremity C_1 of B_1C_1 will be on this second circle. But these two circles intersect only once on the same side of the line AB joining their centres; and, therefore, the vertices C and C_1 coincide, and, consequently, AC coincides with A_1C_1, and BC with B_1C_1. Therefore, the enclosed spaces, or areas (*Lemma* 2, *Introduction*), are equal, *as stated.*



COR. 1.—In pairs of right-angled triangles the same conclusions hold good. One pair of angles, the right angles, being always equal, the conditions for equality of area are reduced to two, viz., a side and an acute angle, or any two sides equal in the two triangles. The proof ends by showing that in all such cases, the three pairs of sides of the triangles are respectively equal.

2. *Two Segments of the Same Circle, or of two equal circles, enclosed by two Equal Arcs and their Chords are equal in area.*

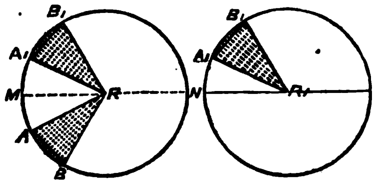
Let the segments ACB and $A_1C_1B_1$ be first taken in the circle R, their chords being AB and A_1B_1. Then, if a diameter MRN be supposed drawn, bisecting the arc AA_1; and if the semicircle MC_1N be turned round the diameter MN so that it fall on the side of the semicircle MCN, according to the remarks in the Introduction to the second chapter, the arc $A_1C_1B_1$ should coincide with the arc ACB throughout. The chords AB and A_1B_1 should also coincide; and therefore, the boundaries of the segments thus coinciding, their areas are equal, *as stated.*



When the segments are one in each of two equal circles R and R_1, as in the diagram, the circle R_1 can be supposed placed over the circle R by making their centres coincide, when consequently their circumferences will coincide; and then the segment $A_1C_1B_1$ of circle R_1 becomes a segment equal to ACB of circle R, and is therefore equal to it in area, as proved above.

3. *Two Sectors of the Same Circle, or of two equal circles, the Central Angles of which are Equal, have equal areas.*

Let the sectors ARB and $A_1R_1B_1$ in the circle R first be taken. Then, since their central angles ARB and A_1RB_1 are equal, the arcs AB and A_1B_1 are (*Theor.* 11, *Chap. II., converse*) equal; and therefore, the segment enclosed by arc AB and chord AB is equal in area to the segment enclosed by the arc A_1B_1 and the chord A_1B_1, as proved (*Theor.* 2).

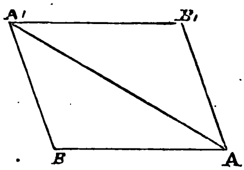


Further, in the two triangles ARB and A_1RB_1, the angles at R are, by supposition, equal, and the sides of the triangles are equal, all being radii. Therefore (*Theor.* 1) their areas are equal;

and also the areas of the sectors, which are the sums of the areas of the segments and of the triangles, are equal, *as stated.*

In the case of the sectors being one in each of two equal circles, the proof is the same. The segments are equal in area, as proved in the last theorem; and the triangles ARB and A_1RB_1, having equal sides and contained angles, as above proved, are equal also in area.

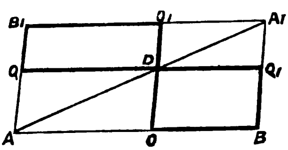
4. *The Diagonal of a Parallelogram divides it into Two Triangles, the areas of which are equal.*



It has been proved above (*Theor.* 1) that two triangles, whose three sides are equal respectively, have equal areas. Now, in the two triangles ABA_1 and AB_1A_1, the side AA_1 is common; the opposite sides AB and A_1B_1 are equal, and also the other pair of opposite sides, AB_1 and A_1B, are equal. Therefore the triangles, having their three sides respectively equal, have (*Theor.* 1) their areas equal, *as stated.*

COR. 1. Squares and rectangles being parallelograms, the general proposition holds good—a diagonal divides them into two right-angled triangles of equal areas.

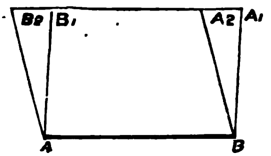
5. *In a Parallelogram the Complements of the parallelograms about the Diagonal are equal in area.*



Let ABA_1B_1 be the parallelogram and AA_1 the diagonal, and OO_1, QQ_1 parallels through a point D on the diagonal to the sides. The parallelogram is thus divided into two parallelograms, AD and A_1D, about the diagonal, and the two parallelograms OQ_1 and O_1Q. But (*Theor.* 4) the triangles AA_1B and AA_1B_1 are equal in area. Also, in the parallelograms about the diagonal, the triangle AOD is equal to the triangle AQD in area; and so is the triangle A_1O_1D equal to A_1Q_1D. Take away respectively these triangles from the triangles AA_1B and AA_1B_1, and the remainders, the parallelograms OQ_1 and O_1Q, which are the complementary parallelograms, are (*Ax.* 3) equal in area, *as stated.*

6. *Parallelograms on the Same Base and between the Same Parallels are equal in area.*

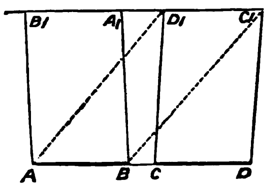
Let AB be the common base, and ABA_1B_1 and ABA_2B_2 two such parallelograms on AB and between the parallels AB and A_1B_2.



Then, since AB_1 is parallel to A_1B, and also AB_2 parallel to A_2B, the angles between these parallels, A_1BA_2 and B_1AB_2, are equal. Also the sides AB_1 and BA_1 are equal, and AB_2 and BA_2 also equal. The two triangles AB_1B_2 and BA_1A_2 are therefore (*Theor.* 1) equal in area. Take these equal triangles from ABA_1B_2 in succession, and the remainders, the parallelograms AA_1 and AA_2, are (*Ax.* 3) equal in area, *as stated.*

7. *Parallelograms on Equal Bases and between the Same Parallels are equal in area.*

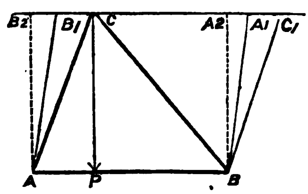
Let AB and CD be the equal bases, and ABA_1B_1 and CDC_1D_1 the two parallelograms on these bases and between the parallels AD and B_1C_1.



Join A with D_1 and B with C_1. Then (*last Theor.*) the parallelograms AA_1 and AC_1 are equal, the line D_1C_1, which is equal to CD, being equal to AB. Also, the parallelogram AC_1, being on the base D_1C_1, is equal to parallelogram CC_1 on the same base and between the same parallels AD and B_1C_1. Therefore, the parallelograms AA_1 and CC_1, being each equal to parallelogram AC_1, are (*Ax.* 1) equal to each other, *as stated.*

8. *If a Parallelogram and a Triangle be on the Same Base, and between the Same Parallels, the area of the triangle is half that of the parallelogram.*

Let ABA_1B_1 be the parallelogram, and ACB the triangle, both on the common base AB.



Let, also, BC_1 be a parallel through B to AC meeting B_1A_1 produced at C_1. Then ABC_1C is a parallelogram; and, BC being one of its diagonals, it divides this parallelogram into two triangles, ABC and BCC_1, equal in area. Therefore the triangle ACB is half the parallelogram AC_1. But (*Theor.* 6) the parallelogram AC_1 is equal to the parallelogram

AA_1. Therefore the triangle ACB is equal in area to half the parallelogram AA_1, *as stated.*

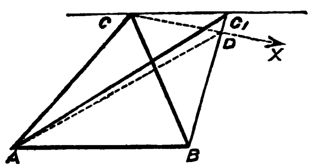
COR. 1. If the parallelogram AA_1 become a rectangle, ABA_2B_2, the sides, AB_2 and BA_2 are both perpendicular to AB; and also parallel and equal to the perpendicular CP from the vertex C of the triangle on its base AB. The area of a triangle is therefore half the rectangle under its base and the perpendicular from its vertical angle on the base.

9. *Triangles on the Same Base, or on Equal Bases, and between the Same Parallels, are equal in area.*

This follows immediately from the last three theorems. Triangles (*last Theor.*) are the halves of parallelograms on the same base or on equal bases and between the same parallels; which parallelograms are (*Theors.* 6 *and* 7) proved equal. And therefore their halves, the triangles, are equal in area, *as stated.*

10. *If Two Triangles stand on a Common Base and their Areas are Equal, the line joining their vertices is parallel to the common base.*

Let the two triangles be ACB and AC_1B on the common base AB; and suppose, if possible, that the line CC_1 joining their vertices is not parallel. Then some other line through C, the directive X must be parallel. Join the point D in which X cuts BC_1 with A. Then, since CD, or X, is by supposition parallel to AB, the triangles ACB and ADB are (*last Theor.*) equal in area. But the triangles ACB and AC_1B are given equal. Therefore, the triangles AC_1B and ADB are equal, which is impossible (*Ax.* 10); and CC_1, not X, is the parallel to AB, *as stated.*



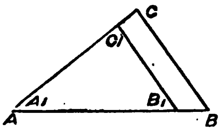
If the two triangles be on two equal bases instead of a common base, the demonstration is the same.

COR. 1. Hence; if any number of triangles be constructed on a common base having all the same area, their vertices will be on a directive parallel to the base, which is the opposite side, produced into a directive, of a rectangle constructed on the base of double the given area. This directive, from its being

"the place" where these vertices lie, is termed the *locus* of the vertices.

11. *If Two Equiangular Triangles have Equal Areas, the sides opposite the equal angles are respectively equal.*

Let ABC and $A_1B_1C_1$ be the two equiangular triangles; and suppose the vertex A_1 of the triangle $A_1B_1C_1$ made to coincide with the vertex A of ABC, and the side A_1B_1 to fall on AB. Then, since the angles A_1 and A are equal, the side A_1C_1 must fall on AC. But wherever the extremities B_1 and C_1 may fall on AB and AC, since the angles B_1 and C_1 are equal to the angles B and C, the line B_1C_1 joining these extremities must either be parallel to BC or coincide with it. It cannot be parallel, for then the areas of the two triangles could not be equal, as is supposed. It must therefore coincide with BC, and consequently the pairs of sides opposite equal angles be respectively equal, *as stated*.



12. *The Areas of Two Squares on Equal Lines are equal.*

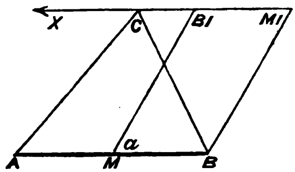
For each square is divided by its diagonal into two isosceles right-angled triangles, which are its halves. Two of these isosceles triangles, one taken from each square, having, by the statement, two sides in one equal to two sides in the other and the contained angles (right angles) equal, their areas (*Theor.* 1) are equal. Therefore their doubles, the squares, are equal in area, *as stated.*

CONVERSELY.—*If two squares have equal areas their sides are equal.* —This follows from the last theorem. For each square may be divided by a diagonal into two equal right-angled isosceles triangles; and these, being equiangular triangles, all having their areas equal, they must (*last Theor.*) have their sides equal. The sides of the squares are therefore equal, *as stated.*

PROBLEMS.

1. *To construct a parallelogram equal to a given triangle ACB, and having an angle a of a given magnitude.*

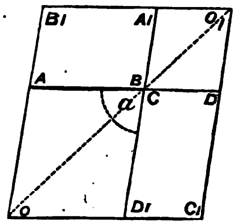
Through the vertex C of the triangle draw a directive X parallel to AB. Bisect AB in M, and from M draw MB_1 making with MB the angle BMB_1 equal to a to meet the directive X. Through B draw BM_1 parallel to MB_1 also meeting X. The parallelogram MM_1 is that required.



For, the parallelogram MM_1, being between the same parallels with the triangle ACB but on a base MB half of the base AB, is (*Theor.* 6) equal in area to the triangle ACB; and also its angle at M is equal to a; *as required.*

2. *To construct on a given line AB a parallelogram, the area of which shall be equal to that of a given triangle, and which shall have an angle a of a given magnitude.*

By the last problem construct a parallelogram of the area of the given triangle having one of its angles of the given magnitude. Produce the given line AB so that the produced part CD be equal to one of the sides of the constructed parallelogram; and let CD_1 be drawn, making the given angle a with AB, and equal to the other side of the constructed parallelogram. Draw, then, through D and D_1 the lines DC_1 and D_1C_1 parallel to CD_1 and CD meeting in C_1. Produce C_1D_1 to meet in O a parallel through A to CD_1. Join O with C and produce OC to meet at O_1, C_1D also produced. From O_1 draw a parallel $O_1A_1B_1$ to AB to meet OA produced to B_1. The parallelogram ABA_1B_1 is the one required.

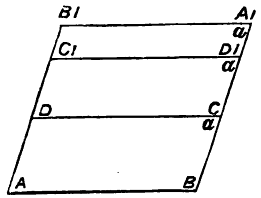


For, the figure OC_1O_1B is, by the above construction, a parallelogram; and the parallelograms AA_1 and CC_1, being the complements of the parallelograms OC, O_1C about the diagonal OO_1, are equal (*Theor.* 5) in area. But the parallelogram CC_1 by construction is equal to the given triangle. Therefore parallelogram

AA_1 constructed on AB, with its angle A equal to the alternate angle a, is also equal to the given triangle, *as required.*

3. *To construct a parallelogram equal in area to a given right-lined figure, and having an angle a of a given magnitude.*

Let the given right-lined figure be divided into triangles; and construct (*Prob.* 1) a parallelogram $ABCD$ with a for one of its angles, and having an area equal to that of one of the triangles. On the side CD of this parallelogram construct (*Prob.* 2) another CDC_1D_1, also having a for an angle and equal to a second triangle. In like manner construct parallelogram $C_1D_1A_1B_1$ with angle a, and equal to a third triangle; and so on with all the triangles if there be more. The figure ABA_1B_1 is then the parallelogram required.



For the angles marked a in the diagram are by construction equal. Then, since in the parallelogram DD_1 the sum of the angles a and C is (*Theor.* 9, *Chap. I.*) equal to two right angles, the sum of the a of the parallelogram AC and the adjacent angle C of parallelogram DD_1, is two right angles. Therefore (*Theor.* 4, *Chap. I.*) the sides BC and CD_1 coincide in direction and form a continuous line. In like manner it is proved that the sides CD_1 and D_1A_1 coincide in direction. Therefore the three sides BC, CD_1, D_1A_1 form one line. Similarly, the sides AD, DC_1 and C_1B_1 form one continuous line. The four sides AB, CD, C_1D_1, and A_1B_1 moreover, are by construction equal and parallel to each other, and therefore A_1B_1 is equal and parallel to AB, and the figure ABA_1B_1 is a parallelogram which has an area equal to that of the given right-lined figure, *as required.*

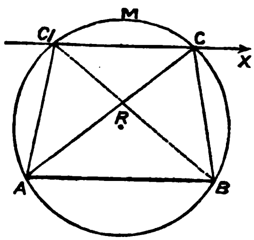
COR. 1. Hence a parallelogram with a given angle a may be constructed on a given line equal in area to a given right-lined figure, by constructing (*by this Prob.*) any parallelogram having that angle and area, and then on the given line constructing another parallelogram (*last Prob.*) of the same angle and area.

COR. 2. A parallelogram may also be constructed on a given line equal to the sum or difference of two right-lined figures by constructing the equivalent parallelograms, in the case of the sum, one above the other, so as to form one parallelogram ; and, in the case of the difference, constructing the

parallelograms equivalent to the triangles of the figure of greater area over each other upwards, and those of the less figure, in a reverse order, downwards, from the upper side of the greater equivalent parallelogram. Thus the difference becomes constructed on the given line.

4. *Given the base AB and the vertical angle of a triangle, and its area, to construct it.*

Let AB be the base. Construct on it (*Prob.* 15, *Chap. II.*) a segment of a circle containing an angle equal to that given. Construct, then, on AB a rectangle of double the given area of the required triangle; and let the directive X be the upper side, indefinitely produced, of this rectangle, cutting the circle in the points C and C_1. Join these points with A and B. Either of the triangles ACB, AC_1B so formed is the required triangle.



For both triangles stand on the given base AB; and, their vertices being on the segment AMB constructed on AB, their vertical angles C and C_1 are equal to the given angle. Their vertices being also on the directive X parallel to AB, their areas (*Theors.* 8 & 9) are equal to each other and to the given area.

N.B.—If the directive touch the circle at its highest point M, the triangle will be the greatest possible in area that can be constructed on AB with the given vertical angle. If X do not cut the circle but lie above it, the solution is impossible. It should also be observed that, in the general solution, the two triangles, ACB and AC_1B, are the same in sides and angles, but reversed.

EXERCISES.

1. Given base and area of a triangle, and that its vertex be on a given directive, construct the triangle.

2. Given base and area of a triangle, construct it so that its vertex be on a given circle. How many solutions are there? And when is the problem impossible?

3. In the last question when is the area the greatest possible and when the least possible, the base and circle only being given?

4. If two triangles have a common base and stand on opposite sides of it, and have their areas equal, the line joining their vertices is bisected by the common base.

5. Inscribe a parallelogram of a given area in a triangle, its sides being parallel to two sides of the triangle.

6. When will the area of this parallelogram be the greatest that can be so inscribed?

7. How inscribe a *rectangle* in a triangle, its area being given; and when will the area be the greatest possible?

8. Inscribe in a circle a rectangle of a given area. When will the area be the greatest possible?

9. Two sides of a triangle being given, when is the area the greatest possible?

10. Bisect the area of a triangle from a point on one of its sides.

11. Trisect the area of a triangle by lines drawn from a point within it.

12. If two triangles stand on two adjacent sides of a parallelogram and have a common vertex on the diagonal through the meeting of the sides, their areas are equal.

13. If two triangles on given bases, and having a common vertex, have the sum or difference of their areas constant, their common vertex in each case is on a directive. How determine these directives?

14. If, with any point as a common vertex within a parallelogram, three triangles be constructed standing, two of them on two adjacent sides of the parallelogram and the third on the diagonal through the meeting of these sides, the area of that on the diagonal is equal to the difference of those on the sides.

15. If the point be taken without the parallelogram, but within the vertically opposite angles of the parallelogram at which the sides meet, the triangle on the diagonal is still equal to the difference between those on the adjacent sides.

16. If the point be taken without the parallelogram, but within the two angles supplementary to that at which the sides meet, the area of the triangle on the diagonal is equal to the sum of the areas of those on the two adjacent sides.

17. Prove that the triangle of greatest area that can be inscribed in a circle is the equilateral triangle.

CHAPTER V.

RECTANGLES AND SQUARES.

RECTANGLES and squares have next to be considered in connection with their areas, as constructed on lines which are the sums or differences of two or more lines or single lines divided into segments.

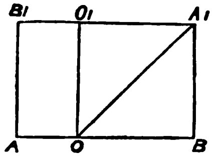
The following proposition is practically useful in both cases:—

1. *The Half Sum of two lines With their Half Difference is equal to the greater line, and the half sum Less by the half difference is equal to the less line.*

Let AO, A_1O be the two lines placed *in directum*, and consequently AA_1 their sum. Then, on taking AO_1 equal to A_1O, the line OO_1 is their difference. Let also M be the bisection of AA_1; then M is evidently also the bisection of OO_1. Therefore, MA is half the sum of AO and A_1O; and MO, the half of OO_1, is half the difference of AO and A_1O. But the half sum AM, with the half difference MO, is the greater line AO; and the half sum A_1M, less by the half difference MO, is the less line A_1O, *as stated.*

2. *The Rectangle under the Sum of Two Lines and Either Line is equal to the square of that line together with the rectangle under the two lines.*

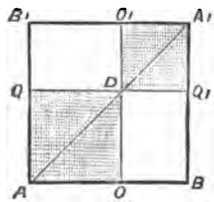
Let AO and BO be the lines *in directum*, and therefore AB their sum, and ABA_1B_1 the rectangle described on AB with A_1B, equal to BO, for its adjacent side; and let, also, OO_1 be a line through O parallel to AB_1, terminating in A_1B_1. The rectangle AA_1 is thus divided into two rectangles, AO_1 and OA_1. But since A_1B is, by construction, equal to BO, the rectangle OA_1 is a square—the square of BO. Also, since OO_1 is (*Theor.* 14, *Chap. III.*) equal to A_1B, it is equal to BO; and therefore the rectangle AO_1 is equal to the rectangle under AO and BO. Hence the rectangle under the sum AB and the line BO is equal to the square of BO, together with the rectangle under AO and BO, *as stated.*



If the rectangle under the sum AB and AO were taken, the result would be similar; the rectangle under AB and AO would be equal to the square of AO with the rectangle under AO and BO.

3. *The Square of the Sum of Two Lines is equal to the sum of their squares with double the rectangle under them.*

Let AB be the sum of the two lines AO and BO, and ABA_1B_1 the square on AB. Also, let OO_1 be a parallel to AB_1 through O cutting the diagonal AA_1 in D, and QQ_1 the parallel through D to AB. Then (*Theor.* 20, *Chap. III.*) the square on AB is divided into two squares, AD and A_1D, about the diagonal, which are the squares of AO and BO, and the two complementary rectangles OQ_1 and O_1Q, whose sides are equal to each other and equal to AO and BO. The area of the square, therefore, is composed of the squares of the lines AO and BO, and double the rectangle under AO and BO, *as stated.*

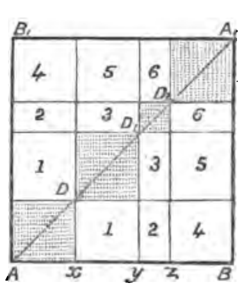


COR. 1. If AO be equal to BO, the line AB is bisected at O; and the squares about the diagonals are each a square of the half line. The two complementary rectangles also are each a square of the half line. Therefore, the square of a line is four times the square of its half.

The principle in the above proposition may be extended.

4. *The Square of the Sum of any Number of Lines is equal to the sum of the squares of the lines, with double the sum of the rectangles under them in pairs.*

Let AB be the sum composed of the lines Ax, xy, yz, and zB, and ABA_1B_1 the square on AB. Then, on drawing parallels through the points x, y, z, to AB_1 or A_1B, intersecting the diagonals in D, D_1, D_2; and through D, D_1, D_2, other parallels to AB or A_1B_1; these parallels divide the whole square into squares (the shaded figures) about the diagonal AA_1, and pairs of equal rectangles (*Theor.* 21, *Chap. III.*) marked 1, 2, 3, 4, 5, 6. The squares about the diagonals are (*same Theor.*) the squares of Ax, xy, yz, and zB; and the complementary rectangles, in pairs, as marked, have their sides equal to the sides of the squares, Ax, xy, yz, and zB. Therefore the area of the square of the sum AB is equal to the sum of the areas of the squares of the lines and of double the sum of the areas of the rectangles under every pair of them.



COR. 1. If the lines Ax, xy, yz, zB be equal, the complementary rectangles will be also squares of the component lines, and every figure into which the square of AB is divided will be a square each equal to another; and the whole square will be as many times the square of the part as is represented by the arithmetical square of the numbers of parts.

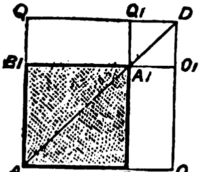
Thus, in the figure, the parts being four, the square of a line is sixteen times the square of its fourth part.

So, also, the square of a line is twenty-five times the square of its fifth part.

The same for other numbers.

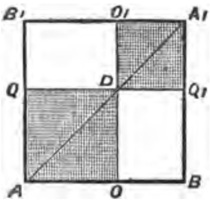
5. *The Square of the Difference of Two Lines is equal to the sum of their squares, less by two rectangles under the lines.*

Let AO and BO be the lines, and therefore AB their difference. Let, also, $AODQ$ be the square on the greater line AO. Let BQ_1 be a parallel to AQ meeting AD in A_1; and through A_1 let B_1O_1 be a parallel to AO. Then, since the rectangle AD is a square, the rectangle AA_1 about the diagonal AA_1 is also a square, and the square of the difference AB. But the square AA_1 is less than the square AD of the greater line AO by the gnomon QDO. It is therefore less than the sum of the squares AD and A_1D by the gnomon and the square A_1D. But the gnomon and the square A_1D together are equal to the two equal rectangles QO_1 and Q_1O. But the sides of these rectangles are equal respectively to AO and BO. Therefore the square of AB is equal to the sum of the squares of AO and BO, less by two rectangles under AO and BO, *as stated.*



6. *The Difference of the Squares of Two Lines is equal to the rectangle under their sum and difference.*

Let AO and BO be two lines forming a sum AB, and ABA_1B_1 the square on AB. Let, also, OO_1 be a parallel to AB_1 through O, cutting the diagonal AA_1 in D, and QQ_1 a parallel to AB, also through D. Then the rectangles AD, A_1D about the diagonal AA_1 are squares, the squares of AO and BO, and the complementary rectangles OQ_1 and O_1Q, have (*Theor.* 20, *Chap. III.*) equal sides equal to AO and BO, and (*Theor.* 5, *Chap. IV.*) also equal areas.

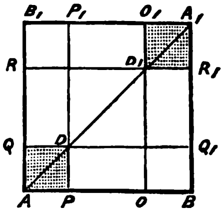


Now, the difference of the squares AD and A_1D cannot be altered by adding to AD the rectangle QO_1 and to A_1D the equal rectangle OQ_1. The difference of the squares AD and A_1D thus becomes the difference of the rectangles AO_1 and BQ_1. But these two rectangles have OO_1 for a common side, which is equal to AB. Therefore the difference of the squares of AO and BO is equal to a rectangle under AB, the sum of AO and BO, and a line equal to the difference of AO and BO, that is, to the rectangle under the sum and difference of the two lines, *as stated.*

7. *The Sum of the Squares of Two Lines with the Difference of their Squares is equal to double the square of*

*the greater line; and the sum of the squares less by the difference is equal to double the square of the less line.**

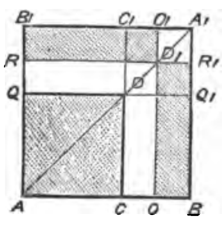
Let AO and BO be the two lines, and AP equal to BO. Also, let ABA_1B_1 be the square on AB_1, and OO_1, PP_1 two parallels to AB_1, cutting the diagonal AA_1 in D_1 and D; and QQ_1 and RR_1 two parallels through D and D_1 to AB. Then, the square AD_1 about the diagonal AA_1 is the square of AO, and the square A_1D_1 is the square of BO. Also, since AP is equal to BO, the square AD is a square of BO. Hence the gnomon RD_1OPDQ is equal to the difference of the squares of AO and BO. Therefore, if the sum of the squares AD_1 and A_1D_1 be added to the gnomon, the square A_1D_1 (equal to the square AD) with the gnomon makes one square of AO, and the other square AD_1 is a second square of AO, that is, the sum of the squares with the difference of the squares is two squares of the greater line, *as stated*.



Further, if the gnomon be taken from the sum of the squares AD_1 and A_1D_1, there will evidently remain the two squares AD and A_1D_1, that is, two squares of the less line BO, *as stated.*

8. *The Sum of the Squares of the Sum and Difference of Two Lines is double the sum of the squares of the lines.*

Let AB be the sum of two lines AO, BO; and suppose OC taken on AO equal to BO. Then AC is their difference. Let, also, ABA_1B_1 be the square on the sum AB, and CC_1 and OO_1 parallels through C and O to AB_1, meeting the diagonal AA_1 in D and D_1, and QQ_1 and RR_1 parallels to AB through D and D_1. Then, the square of AB is (*Theor.* 3) equal to the squares AD_1 and A_1D_1, with the two shaded rectangles OR_1 and O_1R. But, since CO is equal to BO, the two shaded rectangles are equal to the rectangles CD_1 and QD_1, that is, to the gnomon RD_1O with the square DD_1, or to the gnomon with the square A_1D_1. The square of the sum, therefore, of AO and BO, with the square of their difference, is equal



* *See* Appendix, Note 4.

to the sum of one square AD_1, two squares A_1D_1, the gnomon RD_1O, and the square AD. But the square AD with the gnomon is the square AD_1; therefore, the sum of the squares of AB and AC is equal to two squares AD_1 with two squares A_1D_1; that is, to double the sum of the squares of AO and BO; *as stated.*

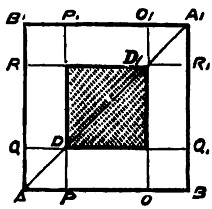
This theorem may otherwise be thus proved from the *third* and *fifth* theorems. The square of the sum is equal to the sum of the squares with double the rectangle under them, and the square of the difference is equal to the sum of the squares less by two rectangles. The excess of the two rectangles in the former case, in a geometric addition, is balanced by the default of the same two rectangles in the latter; and the result is, double the sum of the squares.

By halving the magnitudes in this theorem we may state it differently. The sum of the squares of the lines becomes half the square of the sum with half the square of the difference. But half the square of any line is double the square of its half. Therefore, *the sum of the squares of two lines is double the square of half their sum with double the square of half their difference.*

COR. 1. Hence it follows that, when the sum of two lines is given, the sum of their squares is the least possible when the lines are equal. For, since the sum of their squares is double the square of half their sum (which is a fixed magnitude) with double the square of half their difference, when the difference is nothing the sum of the squares must be least, and the lines equal.

9. *The Difference of the Squares of the Sum and Difference of Two Lines is equal to four rectangles under the lines.*

Let AO and BO be the lines, and ABA_1B_1 the square on their sum AB. Let AP be equal to the less line BO, and OO_1, PP_1 parallels through O and P to AB_1 meeting the diagonal AA_1 in D_1 and D, and QQ_1, RR_1 parallels through D and D_1 to AB. Then, since AP is equal to BO, OP is the difference of AO and BO, and the square DD_1 about the diagonal is the square of the difference of the lines AO, BO. But the square on AB exceeds the square DD_1 by the unshaded marginal figure composed of the four rectangles OQ, QP_1, P_1R_1, and R_1O. But these are (*Theor.* 20,



Chap. III.) four rectangles under AO and BO. Therefore the square of the sum AO exceeds the square of the difference PO by four rectangles under AO and BO, *as stated.*

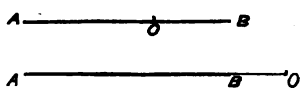
This theorem may otherwise be thus proved from the *third* and *fifth* theorems. The square of the sum is equal to the sum of the squares with two rectangles under them; and the square of the difference is equal to the sum of the squares less by two rectangles. The square of the sum then exceeds the sum of the squares by two rectangles; and the sum of the squares exceeds the square of the difference also by two rectangles. The square of the sum, therefore, exceeds the square of the difference by four right angles.

By taking the fourth parts of all the magnitudes in this theorem we may state it differently. *The rectangle under two lines is equal to the difference of the squares of half their sum and half their difference.*

Cor. 1. Hence it follows that, the perimeter of a rectangle being given, the area is the greatest when the rectangle is a square. For, the perimeter being given, the sum of its two adjacent sides, the half perimeter, is given; and the rectangle under these sides is equal (as above proved) to the difference of the squares of half their sum and half their difference. But the sum of sides being fixed in magnitude, the rectangle is greatest when the square of the difference is least, that is, when it is nothing, or when the sides are equal and the rectangle a square.

Cor. 2. Hence, also, it follows that, of all rectangles having a given area, the square has the least perimeter. For, in any other rectangle of the same area, that area is equal to the difference of the squares of the half sum and half difference of its sides. The square of the half sum must therefore be greater than the square of given area, and consequently the half sum of sides itself greater than the side of that square. The perimeter of the square, therefore, is least.

We next consider the areas of rectangles and squares in connection with lines divided into segments equal or unequal, or into both.



The first object is a clear account of a segment.

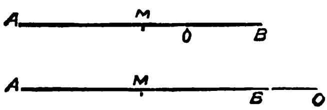
Suppose the line AB (in the upper figure) divided into two parts at O. Then AO and BO are manifestly the segments of AB. But the term "segment" may carry a larger meaning; and how are we to define it so as to convey a distinct general idea?

The account generally given is, that the segments of a line, such as AB, are the distances of the point of section O from the extremities A, B, of the line; and this holds good for the popular idea of "cutting a line."

But Modern Geometry has extended the idea of a segment to cases in which the point of section O is outside the line AB, either to right or left, but on its direction. Thus, when O is taken to the right of B on AB produced (as in the lower figure), the distances AO and BO of O from A and B are still considered "segments" of AB; but the line AB is then said to be cut *externally*. The line may likewise be cut externally to the left of A.

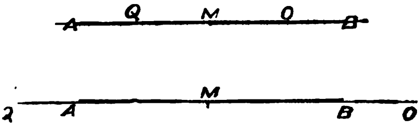
The student will notice that, in internal section, the segments are measured from O in opposite directions, from right to left for AO, and left to right for BO; whereas, in external section they are both from right to left or left to right.

It should also be noted that when the point of section is either extremity of AB, one of the segments is AB itself, and the other vanishes. Also, when AB is bisected at M, the segments AM, BM are equal; but it cannot be bisected externally within any finite distance; AO being always greater than BO. These external segments, however, tend to equality, as the distances of O from A and B increase in magnitude.



When a line is both bisected and cut, either internally or externally, into unequal segments, there are altogether five segments to be considered in each case, namely, the unequal parts AO and BO, the equal parts AM and

BM, and the intermediate part *MO*. In internal section the intermediate part is less than the half line. In external section it is greater.



In *internal* section the sum of the segments *AO* and *BO* (as in the upper figure of the adjoining diagram) is the line *AB*; or, if *M* be the bisection of *AB*, is double *AM*, the half line. *AM or BM is then half the sum of the segments.* Also, if *AQ* be taken equal to *BO*, then *QO* will be the difference of the segments, and *M* will be also the bisection of *QO*. Therefore *MO is half the difference of the segments.*

In *external* section the difference of the segments *AO*, *BO* (as in the lower figure of the same diagram) is the line *AB*; or, if *M* be the bisection of *AB*, is double *AM*, the half line. *AM or BM is then half the difference of the segments.* Also, if *AQ* be taken to the left from *A* equal to *BO*, then *QO* will be the sum of the segments; and *M* will also be the bisection of *QO*. Therefore, *MO is half the sum of the segments.*

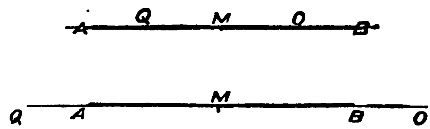
The following principle, which places this subject in another point of view, and will be applied to some of the immediately following theorems, is worth knowing. It may be termed—

THE PRINCIPLE OF INTERNAL AND EXTERNAL CONVERSION.

The relations of magnitude of the segments of a line bisected and cut externally may be reduced to those of a line bisected and cut internally; and, *vice*

versâ, the relations of magnitude of the segments of a line bisected and cut internally may be reduced to those of external section.

Let AB (in the lower figure of the diagram) be bisected at M and cut externally at O. Then, if AQ be taken to the left from A equal to the external segment, BO, the line QO becomes the sum of the segments AO and BO, and is divided at A internally into segments AO, AQ, equal to the external segments, AO, BO, of AB. The point M, the bisection of AB, becomes thus also the bisection of QO, and MO and MQ become each the half sum of QA and OA, and, therefore, of AO and BO. Also, AM or BM becomes the half difference of OA and QA, and therefore of AO and BO. The relations, therefore, between AO, BO, MO, and MA are those of OA, QA, MO, and MA of QO, cut internally at A.

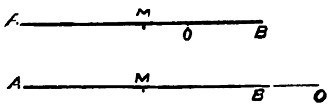


Similarly, by taking AQ to the right of A (as in the upper figure), equal to BO in the case of a line AB, bisected at M and cut again internally at O, the lines QB and OB become the external segments of QO cut externally; and MA and MB become each their half sum and MQ and MO each their half difference.

The following theorems of Euclid on divided lines immediately follow from the principles so far established. The mode of statement and the terminology are altered.

10. *If a Line be Bisected and Cut again Internally, the Rectangle under the Unequal Segments is equal to the difference of the squares of the half line and intermediate segment.* (EUCLID, Book II. 5.)

Let AB be the line (upper figure) bisected at M and cut again internally at O. Then the segments AO and BO are two lines together forming a sum AB, and MA is their half sum and MO their half difference. But (*by Theor.* 9) the rectangle under any two lines is equal to the difference of the



squares of half their sum and half their difference. Therefore the rectangle under AO and BO is equal to the difference of the squares of the half line MA and intermediate segment MO, *as stated.*

This may also be proved from Theorem 6; for, AM and MO being two lines, AO and BO are respectively their sum and difference. Therefore, since, by that theorem, the difference of the squares of two lines is equal to the rectangle under their sum and difference, the difference of the squares of MA and MO is equal to the rectangle under AO and BO.

11. *If a Line be Bisected, and also Cut Externally, the Rectangle under the External Segments is equal to the difference of the squares of the intermediate segment and half line.* (EUCLID, Book II. 6.)

Of this theorem three proofs may be given.

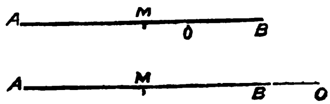
1. It may be reduced by the principle of Internal and External Conversion, above enunciated, to the last theorem.

2. Or proved thus:—The external segments AO, BO (lower figure) being two lines, MO is their half sum, and MA their half difference. But the rectangle under two lines (*Theor.* 9) is equal to the difference of the squares of their half sum and half difference. Therefore, the rectangle under AO and BO is equal to the difference of the squares of the intermediate segment MO, and half line MA, *as stated.*

3. Or proved by Theorem 6. MO and MA being lines, their sum is AO and their difference BO. But the difference of the squares of two lines (*Theor.* 6) is equal to the rectangle under their sum and difference. Therefore, the difference of the squares of MO and MA is equal to the rectangle under AO and BO, *as stated.*

12. *If a Line be Bisected and Cut again Internally, the Sum of the Squares of the Unequal Segments is double the sum of the squares of the half line and intermediate segment.* (EUCLID, Book II., 9.)

Let AB (in the upper diagram) be bisected at M and cut again internally at O. Then, the segments AO and BO are two lines together forming a sum AB, and MA is their half sum, and MO their half difference. But the sum of the



squares of any two lines (*Theor.* 8) is equal to double the sum of

the squares of their half sum and half difference. Therefore, the sum of the squares of AO and BO is equal to double the sum of the squares of the half line MA and intermediate segment MO, *as stated.*

13. *If a Line be Bisected and Cut also Externally, the Sum of the Squares of the External Segments is double the sum of the squares of the intermediate segment and half line.* (EUCLID, Book II., 10.)

Of this theorem two proofs may be given.

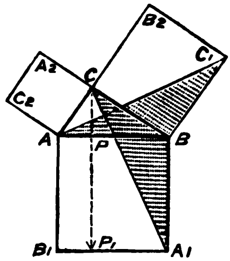
1. It may be reduced by the Principle of Internal and External Conversion, above enunciated, to the last theorem.
2. Or proved thus:—The external segments, AO, BO, being two lines, MO is their half sum, and MA their half difference. But the sum of the squares of any two lines (*Theor.* 8) is equal to double the sum of the squares of their half sum and half difference. Therefore, the sum of the squares of AO and BO is double the sum of the squares of the intermediate segment MO and half line MA, *as stated.*

The next subject is that of squares on the sides of triangles and other lines connected with them, and squares and rectangles connected with circles. We commence with the right-angled triangle, the side of which opposite the right angle is usually termed "the hypothenuse," the other two being simply named "the sides." The celebrated theorem of Pythagoras claims the first attention.

14. *The Square of the Hypothenuse of a Right-Angled Triangle is equal to the sum of the squares of the sides.*

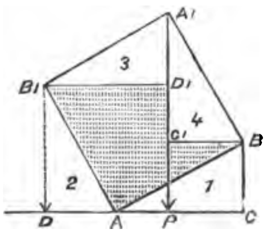
Let ACB be the right-angled triangle, AB its hypothenuse, and AA_1, BB_2, and AA_2 the three squares. Let, also, CPP_1 be the perpendicular from C to both AB and A_1B_1; and AC_1 and A_1C joining lines. Then, the angles ABA_1 and CBC_1, being both right angles, are equal. Add to both the angle ABC; and the angle A_1BC is equal to ABC_1. Also, the sides

A_1B and CB are equal respectively to AB and C_1B. Therefore (*Theor.* 9, *Chap. III.*) the remaining sides AC_1 and A_1C of the triangles ABC_1 and A_1BC are equal, and their areas (*Theor.* 1, *Chap. IV.*) also equal. But the triangle ABC_1, being on the base BC_1, and between the parallels AB_2 and BC_1, is (*Theor.* 8, *Chap. IV.*) half the square BB_2 in area; and A_1BC, being on the base A_1B and between the parallels CP_1 and A_1B, is likewise half the rectangle BP_1. Therefore the square BB_2 is equal in area to the rectangle BP_1, each being double the equal triangles.



In like manner, it may be proved that the square AA_2 is equal to the rectangle AP_1; and, therefore, the sum of the squares AA_2 and BB_2, being equal to the sum of the rectangles AP_1 and BP_1, is equal to the square AA_1, *as stated.*

ANOTHER PROOF.—Let ACB be the right-angled triangle, AB the hypothenuse, and ABA_1B_1 the square on AB. Also, let A_1P and B_1D be perpendiculars to the directive DAC; and B_1D_1 and BC_1 be parallels to DAC. Then, in the triangle marked 3, its sides are parallel to those of triangle 1; and its angles therefore equal to those of 1. In the triangles 2 and 4, their sides being perpendicular to those of 1, their angles (*Theor.* 6, *Chap. I.*) are also respectively equal to the angles of 1. All four triangles, therefore, are right-angled and equiangular. Their hypothenuses, also, are equal, being the sides of the square on AB; and therefore the four triangles are equal altogether. Also, since B_1D is equal to B_1D_1, the rectangle DD_1 is the square of B_1D_1 or of AC; and, since BC_1 is equal to BC, the rectangle CC_1 is the square of BC.



Now, take from the whole figure $DCBA_1B_1$ the triangles 1 and 2; the remainder is the square of the hypothenuse AB. Take, again, the equal triangles 3 and 4 from the same figure, and the remainder is the sum of the two squares of AC and BC. The square of AB is therefore equal in area to the sum of the squares of AC and BC, *as stated.*

By the aid of this demonstration the square on AB can be cut

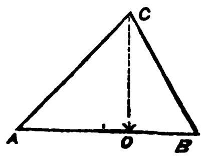
by a scissors so that the triangles 3 and 4, with the shaded figure $AB_1D_1C_1B$, may make up the two squares, DD_1 and CC_1, of the sides; or, *vice versâ*, the two squares, placed side by side in one piece of cardboard, may be cut so into the two triangles 1 and 2 and the shaded figure that, on placing 1 and 2 over 3 and 4, the square of the hypothenuse may be produced.

From the first demonstration of this theorem the following corollaries (equivalent to leading theorems) are derived:—

COR. 1. *The square of either side of a right-angled triangle is equal to the rectangle under the hypothenuse and its segment adjacent to that side made by the perpendicular from the right angle.* For it is there shown, that the square on BC is equal to the rectangle BP_1. But the rectangle BP_1 is the rectangle under A_1B, or AB, and the adjacent segment BP of the hypothenuse, *as stated.*

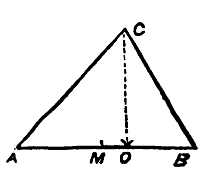
COR. 2. *The square of the perpendicular from the vertex of a right-angled triangle upon the hypothenuse is equal to the rectangle under the segments of the hypothenuse made by the perpendicular.* This is a consequence of the preceding corollary. The squares of CP and BP together are equal to the square of BC. But the square of BC is (as just proved) equal to the rectangle $AB{\cdot}BP$, which (*Theor.* 2) is equal to the square of BP with the rectangle $AP{\cdot}BP$. Take the square of BP from both; and the square of CP remains equal to the rectangle $AP{\cdot}BP$, *as stated.*

COR. 3. *The difference of the squares of the sides of a triangle is equal to the difference of the squares of the segments of the base made by the perpendicular from the vertical angle upon the base.* Let ACB be the triangle and CO the perpendicular. Then, the square of AC is equal to the sum of the squares of AO and CO. Also, the square of BC is equal to the sum of the squares of BO and CO. Therefore, since the square of CO is common to both squares, the difference of the squares of AC and BC is equal to the



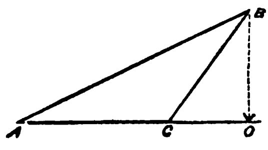
difference of the squares of the segments AO and BO, *as stated.*

15. *The Difference of the Squares of the Sides of a Triangle is equal to Double the Rectangle under the Base and the Segment intercepted between the bisection of the base and the foot of the perpendicular from the vertical angle on the base.*



Let ACB be the triangle, CO the perpendicular from C upon the base AB, and M the bisection of the base. Then, since (*Cor.* 3, *last Theor.*) the difference of the squares of AC and BC is equal to the difference of the squares of AO and BO, it is equal (*Theor.* 6) to the rectangle under the sum and difference of AO and BO. But AB is the sum of AO and BO, and double MO is their difference, as shown in the explanations already given on the sections of lines. The difference of the squares of AO and BO is therefore equal to a rectangle under AB, and double MO, that is, to two rectangles under AB and MO, *as stated.*

16. *The Square of the Side Subtending the Obtuse Angle of a Triangle is equal to the sum of the squares of the two other sides, with double the rectangle under either of these sides, and the external segment of that side made by the perpendicular from the opposite angle.*

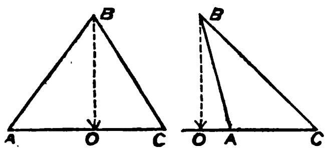


Let ACB be the triangle, AB the side opposite the obtuse angle C, and BO the perpendicular from B on AC, cutting it externally at O. Then, the square of AB is equal to the sum of the squares (*Theor.* 14) of AO and BO. But the square of AO (*Theor.* 3) is equal to the sum of the squares of AC and CO, with double the rectangle under AC and CO. But the squares of BO and CO are together equal to the square of BC,

Hence, the square of AB is equal to the sum of the squares of AC and BC, with double the rectangle under AC and the external segment CO, *as stated.*

17. *The Square of a Side Subtending an Acute Angle of a Triangle is less than the sum of the squares of the other two sides, by double the rectangle under either of these sides and the internal segment of that side made by the perpendicular from the opposite angle.*

Let ACB be the triangle in either figure, and AB the side opposite the acute angle C. Then, the square of AB is equal (*Theor.* 14) to the sum of the squares of BO and AO. But AO is the difference of AC and OC. Its square, therefore, is equal (*Theor.* 5) to the sum of the squares of AC and OC less by two rectangles under AC and OC. But the sum of the squares of BO and CO is the square of BC. Therefore, the square of AB is equal to the sum of the squares of AC and BC less by double the rectangle under AC and the internal segment CO, *as stated.*



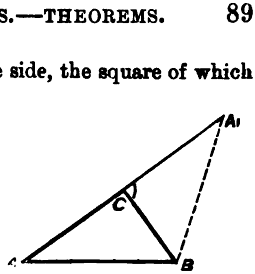
It should here be observed that in the right-hand figure the segment OC is still considered internal, though its extremity O is outside AC; the reason being, that it lies within the angle C, whereas, in Theorem 16, the corresponding segment OC lies outside and away from the angle C of the triangle.

18. *If the Square of One Side of a Triangle be equal to the Sum of the Squares of the other Two Sides, the angle opposite that side is a right angle.*

This follows immediately from the two preceding theorems. For, if the angle be not right, it must be either obtuse or acute. If it be obtuse, the square opposite that angle (*Theor.* 16) must be greater than the sum of the squares of the other two sides, which is contrary to the supposition made. If it be acute, the square opposite that angle must (*Theor.* 17) be less than the sum of the squares of the two other sides, also contrary to supposition. Since, therefore, the angle can be neither obtuse nor acute, it must be a right angle.

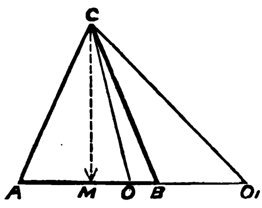
Euclid's direct proof of this proposition is so simple that it is worth being known.

Let ACB be the triangle, and AB the side, the square of which is equal to the sum of those of the other two sides, AC and BC. Let, also, A_1C be equal to AC, but perpendicular to BC at C. Then, since A_1CB is a right angle, the square of A_1B is equal (*Theor.* 14) to the sum of the squares of A_1C and BC. It is therefore, by the construction, equal to the sum of the squares of AC and BC; that is, to the square of AB. But equal squares (*Theor.* 12, *Chap. IV.*, *Converse*) have equal sides; therefore A_1B is equal to AB, and the two triangles ACB and A_1CB have their three sides equal respectively. Therefore the angle opposite AB is (*Theor.* 8, *Chap. III.*) equal to that opposite A_1B. But the latter angle is a right angle. Therefore ACB also is a right angle, *as stated.*



19. *In an Isosceles Triangle the Difference of the Squares of either of the Equal Sides, and Any Line drawn from the Vertex internally or externally to the base, is equal to the rectangle under the segments of the base made by the line drawn.*

Let ACB be the isosceles triangle; CM the perpendicular from the vertex C on the base, bisecting it at M; and CO and CO_1 two lines from C, meeting the base AB, one internally at O and the other externally at O_1. There are thus two cases, internal and external section of AB.



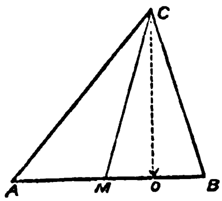
First Case.—The line CO, being within the triangle, the difference of the squares of AC and CO is equal (*Theor.* 14, *Cor.* 3) to the difference of the squares of AM and MO. But this difference of squares is (*Theor.* 6) equal to the rectangle under the sum and difference of AM and MO. But the sum of AM and MO is AO; and their difference is the difference of BM and MO, that is, to BO. Therefore the difference of the squares of AC and CO is equal to the rectangle $AO{\cdot}BO$, *as stated.*

Second Case.—The line CO_1, being without the triangle ACB, it divides the base externally at O_1 into the segments AO_1 and BO_1. But the difference of the squares of CO_1 and AC is equal (*Theor.* 14, *Cor.* 3) to the difference of the squares of MO_1 and AM; that is (*Theor.* 6) to the rectangle under the sum and difference of MO_1 and AM. But the sum of MO_1 and AM is the

segment AO; and the difference of MO_1 and AM is equal to the difference of MO_1 and BM, that is, to the external segment BO_1. The difference of the squares of CO_1 and AC is therefore equal to the rectangle $AO_1 \cdot BO_1$, *as stated.*

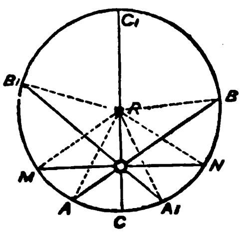
20. *The Sum of the Squares of the Two Sides of a Triangle is equal to double the sum of the squares of the half base and bisector from the vertical angle of the base.*

Let ACB be the triangle; CM the bisector at M of the base AB, and CO the perpendicular from the vertex C on the base. Then, the square of AC opposite the obtuse angle AMC is equal (*Theor.* 16) to the sum of the squares of AM and MC, *with* double the rectangle under AM and MO. In like manner, the angle BMC, being acute, the square of BC is equal (*Theor.* 17) to the sum of the squares of BM and MC, *less* by two rectangles under BM and MO. But BM is equal to AM; and the rectangles $AM \cdot MO$ and $BM \cdot MO$ are equal. Therefore, as the square of AC exceeds the sum of the squares of AM and MC by two rectangles, and the square of BC falls short of the sum of the squares of AM and MC by the same two rectangles, the sum of the squares of AC and BC must be equal to the sum of two squares of AM and two squares of MC, *as stated.*



21. *The Rectangles under the Segments of Two Chords which intersect Within a Circle are equal in area; and each equal to the square of the semichord of their point of intersection.*

Let the chords be AB and A_1B_1, meeting at O; and suppose the centre R of the circle joined with O and with the extremities, A, B, A_1, B_1, of the chords. Then, since ARB is an isosceles triangle, and RO a line drawn from its vertex R to O within its base AB, the rectangle under AO and BO is equal (*Theor.* 19) to the difference of the squares of RA and RO. In like manner, in the isosceles triangle, A_1RB_1, the rectangle $A_1O \cdot B_1O$ is equal to the difference of the squares of RA_1 and RO. But RA and RA_1, being radii, are equal; and the differences of their



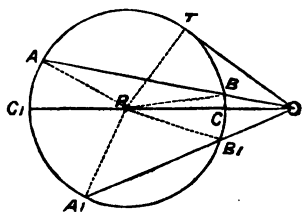
squares and that of RO are equal. Therefore, the rectangles $AO \cdot BO$ and $A_1O \cdot B_1O$ are equal, *as stated.*

These rectangles are, moreover, equal each to the square of the semichord of the point O.* For, if MN be a chord of the circle perpendicular to RO, it is bisected (*Theor.* 2, *Chap. II.*) at O by the perpendicular RO on it from the centre R; and its half, MO or NO, is the semichord of O. But, since the triangle MOR is right-angled at O, the square of MO is (*Theor.* 14) equal to the difference of the squares of MR and RO, that is, to the difference of the squares of the radius and RO. But this difference of squares has been proved equal to the rectangles $AO \cdot BO$ and $A_1O \cdot B_1O$; and therefore both rectangles are equal to the square of the semichord MO, *as stated.*

In the particular case when one of the chords through O becomes the diameter $CORC_1$; since CC_1 is bisected at R and cut unequally at O, the rectangle $CO \cdot C_1O$ under the unequal segments is equal (*Theor.* 10) to the difference of the squares of the half line RC and the intermediate part RO, or of RM and RO, and therefore equal to the square of the semichord MO, *as stated.*

22. *If Two Chords of a Circle Intersect Externally, the rectangles under their segments are equal; and each equal to the square of either tangent to the circle from their point of intersection.*

Let the chords be AB and A_1B_1 meeting externally, or outside the circle, in O; and let the centre R be supposed joined with O and with the points A, B, A_1, B_1. Then, ARB being an isosceles triangle, the rectangle $AO \cdot BO$ is (*Theor.* 19, *Case* 2) equal to the difference of the squares of RO and RA. In like manner, the rectangle $A_1O \cdot B_1O$ is equal to the difference of the squares of RO and RA_1. But RA and RA_1, being radii, are equal; and the differences of the square of RO and their squares are equal. Therefore, the rectangles $AO \cdot BO$ and $A_1O \cdot B_1O$ are equal, *as stated.*



These rectangles are also equal each to the square of the

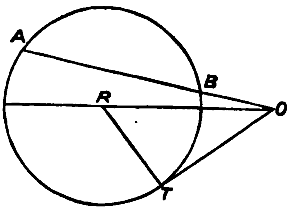
* *See* Appendix, Note 5.

tangent to the circle from O. For, OT being the tangent, and RT the radius to its point of contact, the triangle OTR (*Theor.* 15, *Chap. II.*) is right-angled; and the square of OT, consequently (*Theor.* 14), is equal to the difference of the squares of RO and RT; that is, of RO and the radius RA, or (*Theor.* 19) to the rectangle under AO and BO, *as stated.*

In the particular case when the chord CC_1 through O passes through the centre R, the line CC_1 being bisected at R and cut externally at O, the rectangle $OC{\cdot}OC_1$ is equal (*Theor.* 11) to the difference of the squares of OR and RC, that is, as in the case of the other chords, the difference of the squares of RO and of the radius, that is, to the square of the tangent OT.

23. *If from a Point Outside a Circle two lines be drawn to it, one a Secant and the other meeting but not cutting the circle, so that the square of the meeting line be equal to the rectangle under the segments of the secant, the line so meeting is a Tangent to the circle.*

Let R be the circle, and OBA the secant from the point O, and OT the line meeting but not cutting the circle, and RT a radius. Then, since the rectangle $OA{\cdot}OB$ (*last Theor.*) is equal to the difference of the squares of OR and of the radius, the square of OT must be equal to the difference of the squares of OR and RT; or the square of OR be equal to the sum of the squares of OT and RT. The triangle OTR must therefore (*Theor.* 18) be right-angled at T; and, therefore, OT be a tangent, *as stated.*



24. *If Two Lines so Intersect each other, Internally or Externally, that the Rectangles under their Segments are Equal, the four extremities of the lines lie on a circle.*

Let AB and A_1B_1 be the two lines cutting each other (*as in Fig.* 1) internally, and (*as in Fig.* 2) externally, at O. Also, let MR, M_1R be perpendiculars to AB and A_1B_1 at their points of bisection, M and M_1, meeting in the point R. Suppose, then, R joined with O, and also with the extremities, A, B, A_1, B_1, of AB and A_1B_1. Then, since AB and A_1B_1 are bisected

at M and M_1, and MR and M_1R are perpendiculars to AB and A_1B_1, the triangles ARB and A_1RB_1 are (*Theor.* 3, *Chap. III., Converse*) both isosceles. Hence, the rectangles $AO \cdot BO$ and $A_1O \cdot B_1O$ are equal (*Theor.* 19) respectively to the difference of the squares of AR and OR and A_1R and OR. But these rectangles are, by supposition, equal. Therefore, the difference of the squares of AR and OR and A_1R and OR are equal; and, consequently, AR (which is equal to BR) is equal to A_1R and B_1R. The four lines RA, RB, RA_1 and RB_2 are thus all equal to each other, and become radii of a circle with R as centre, which may be drawn through the four points A, B, A_1, B_1, *as stated.*

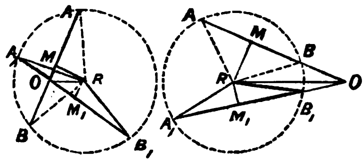
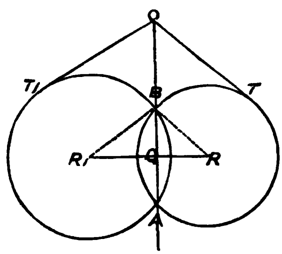


Fig. 1. Fig. 2.

N.B.—The student should observe that this proof applies to both cases of section; but that, in internal section, the rectangles are equal to the difference of the squares of RA and RO; whereas, in external section, they are the difference of the squares of RO and RA, RO being then greater than RA.

25. *The Tangents to two Intersecting Circles from any external Point on their Common Chord are equal; and the common chord divides the line joining their centres into segments, the difference of the squares of which is equal to the difference of the squares of the radii.*

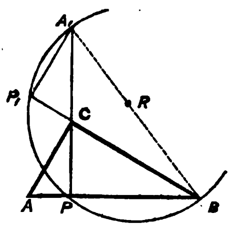
Let R, R_1 be the two circles, and AB the common chord produced to O; and OT and OT_1 the tangents from O. Then, since OT is a tangent to the circle R, its square (*Theor.* 22) is equal to the rectangle $AO \cdot BO$. Also, since OT_1 is a tangent to the circle R_1, its square is equal to the rectangle $AO \cdot BO$. The squares being equal, the tangents OT and OT_1 are therefore equal, *as stated.*



Further, since the chord AB (*Theor.* 6, *Chap. II.*) is perpendicular to RR_1, in the triangle RBR_1, the difference of the squares of the radii RB and R_1B is equal (*Theor.* 14, *Cor.* 3) to the difference of the squares of RQ and R_1Q, the segments of RR_1, *as stated.*

26. *If a Perpendicular be drawn from the vertex of a right-angled triangle to the Hypothenuse, dividing it into segments, the Rectangle under Either Side and the Perpendicular is equal to the rectangle under the other side and the opposite segment of the hypothenuse.*

Let ACB be the right-angled triangle, C being the vertex of the right angle, and CP the perpendicular from C on the hypothenuse AB, dividing it into the segments AP and BP. Then it is required to prove the rectangles $AC \cdot CP$ and $BC \cdot AP$ equal. Let PC be supposed produced, so that CA_1 be equal to CA; and let A_1P_1 be the perpendicular from A_1 on BC produced. Then, since the angles at P and P_1 are right angles, and the legs of each pass through B and A_1, a circle with BA_1 as diameter must pass through P and P_1. But BP_1 and PA_1 are two chords of this circle intersecting at C. Therefore (*Theor.* 21) the rectangle under A_1C and CP is equal to the rectangle under BC and CP_1. But, by construction, rectangle $A_1C \cdot CP$ is equal to rectangle $AC \cdot CP$. Also, since the triangles A_1P_1C and APC are right-angled, and have A_1C equal to AC, and the angle P_1CA_1 (which is equal to BCP vertically opposite) equal to angle PAC, the two triangles are altogether equal; and PA is equal to P_1C. Therefore the rectangle $BC \cdot P_1C$ is equal to BC, AP. But $BC \cdot P_1C$ is already proved equal to $AC \cdot PC$. Therefore the rectangle under AC and CP is equal to the rectangle under BC and AP, *as stated.*



PROBLEMS.

The problems in this chapter are important, having an extensive range of application without recourse to ratio or proportion. The simplest are the three which immediately follow :—

1. *To find a square equal to the sum of two given squares.*

Place AB and AC (*Fig.* 1), the sides of the two given squares, so that they be at right angles to each other at A. Then, on joining B with C, the triangle ABC is right-angled at A ; and therefore (*Theor.* 14) the square of BC is equal to the sum of the squares of AC and AB; and the square on BC is the *required square.*

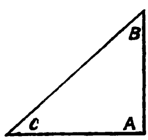


Fig. 1.

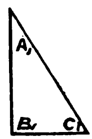


Fig. 2.

COR. 1. Hence a square may be found equal to the sum of any number of given squares : first, by making (*as above*) two of the squares into one; and then making that square and a third into one equal to the sum of the three squares ; and so on, making four squares into one, and five, &c., until all the squares are made into one square.

2. *To find a square equal to the difference of two given squares.*

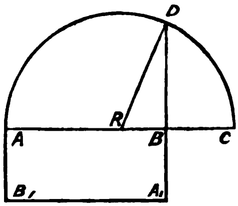
Let A_1B_1, as in Fig. 2 above, be a side of the less given square. Erect, then, at B_1 a directive B_1C_1 perpendicular to A_1B_1 ; and from A_1 draw (*Prob.* 4, *Chap. II.*) a line A_1C_1 to that directive equal to the side of the greater square. Then, since the triangle $A_1B_1C_1$ is right-angled at B_1, the square of A_1C_1 is equal (*Theor.* 14) to the sum of the squares of A_1B_1 and C_1B_1 ; and consequently the difference of the squares of the *given* lines A_1C_1 and A_1B_1 is equal to the square of B_1C_1. The square on B_1C_1 is therefore the *required square*, equal to the difference of the two given squares.

COR. 1. Hence a square may be constructed equal to the sum of any number of given squares less by the sum of any other number of given squares, by making the two sums into two

separate squares by Problem 1; and then, by Problem 2, finding a square equal to the difference of these two squares.

3. *To construct a square equal in area to a given rectilineal figure.*

Construct first (*Prob.* 3, *Chap. IV.*) a rectangle ABA_1B_1 equal to the given figure. Produce AB then to C, so that BC be equal to BA_1, and bisect AC at R. Next, with R as centre and RA as radius, describe a circle. This circle will pass through A and C. Produce A_1B to meet the semicircle on AC at D. The square constructed on BD will be the one required.



For, since AC is bisected at R and cut again internally at B, the rectangle under AB and BC is equal (*Theor.* 10) to the difference of the squares of RA and RB. But RA is equal to RD. Therefore the rectangle $AB \cdot BC$ is equal to the difference of the squares of RD and RB, that is, equal (*Theor.* 14) to the square of BD. But the rectangle $AB \cdot BC$ is equal (*by construction*) to the rectangle AA_1. Therefore the square on BD is equal in area to the given rectilineal figure, *as required.*

The above problems are of importance in connection with those which immediately follow; and, generally, as to all which involve the relations of squares and rectangles described on lines and on their sums and differences, also on the segments of lines divided equally and unequally.

The student, when he reads of the *sums* and *differences* of lines, and of the squares or rectangles constructed on them, and on their sums and differences, is apt to imagine that this addition and subtraction may be performed by a purely arithmetical process, numerical values being attached to the lines. The solutions may thus be had; but they are not *geometrical solutions.* Problems in Pure Geometry are supposed, as the Postulates demand, to be solved by the Rule and Compass; and, to be *truly geometrical* solutions, they must so be solved. For example, the "sum of two squares" means, *geometrically*, a rectangle formed by constructing (*Probs.* 2 *and* 3, *Chap. IV.*) externally on a side of one of the squares a rectangle equal to the other square, the result being a single rectangle equal to their sum; or (*Prob.* 1) making directly a square equal to the sum of the two squares. So, likewise, the *difference* of two squares is the rectangle obtained by constructing on any side of the greater square inwardly into the square a rectangle equal to the less square. The remainder, the

difference of the two rectangles, is a rectangle equal to the difference of the squares. Or the difference may be found by (*Prob.* 2) making directly a square equal to the difference of the two squares. And, geometrically, the same holds good of every other addition of squares and rectangles.

In like manner, the terms "multiplication" and "division" may be used in a *geometrical* sense. The multiplication of two given lines would mean the constructing, by Rule and Compass, a rectangle out of two given lines, the area of which may be considered the geometrical product. And "division" would mean the constructing on a given line, which takes the place of a geometrical "divisor," a rectangle equal to a given area, which becomes the "dividend;" the other side of this rectangle being then the geometrical "quotient." The terms are conventional, but, being based on an analogy, are convenient.

Such is the nature of a geometric solution of a problem. But the student of geometry is not obliged, in an Elementary Treatise on Geometry, actually to perform all these constructions. All that is required is, that he should understand and know them; and be able, should he be called on, actually to solve by Rule and Compass a problem which he has learned. Above all things he should avoid the mistake of considering an arithmetical or algebraical solution geometrical.

The next problems to be considered are those which relate to squares and rectangles on lines, their sums and differences; and also to the squares and rectangles connected with the segments of divided lines.

In the problems on lines, as such, there are five magnitudes involved. They are—

1. The Sum of the lines.
2. Their Difference.
3. The Rectangle under them.
4. The Sum of their Squares.
5. The Difference of their Squares.

These magnitudes, combined in pairs, give the ten problems which next follow.. For sake of simplicity the magnitudes given, when they are areas, are supposed to be given in the form of squares or rectangles. When given in other forms, rectilineal figures of many sides, these figures can be reduced (*Probs.* 2 *and* 3, *Chap. IV.*) to rectangles.

In the problems on Divided Lines, the magnitudes involved are the segments, internal or external, and the half line and intermediate part, as already explained. The constructions, therefore, to be made in the ten problems mentioned apply to these so far

as they correspond. But, in the cases of divided lines, one of the data is the line itself, which is either the sum or difference of the segments, according as the line is cut internally or externally. Hence, any of the ten problems which do not involve either a sum or a difference of lines are inapplicable to the cases of *divided* lines. This reduces the problems of divided lines to six, viz.:—

1 and 2. To cut a line, internally or externally, so that the rectangle under its segments be a given magnitude.

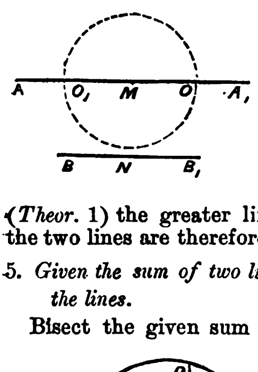
3 and 4. To cut a line, internally or externally, so that the sum of the squares of the segments be a given magnitude.

5 and 6. To cut a line, internally or externally, so that the difference of the squares of the segments be a given magnitude.

Each of these six problems will be considered under the heading of the problem of the ten to which it corresponds.

4. *Given the sum and difference of two lines, to find the lines.*

Let AA_1 and BB_1 be the given sum and difference of the lines.

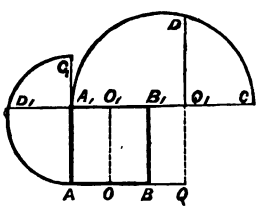


Bisect AA_1 at M and BB_1 at N. Then AM is half the given sum, and BN half the given difference. With M, then, as centre, and a distance MO, equal to BN as radius (*Prob.* 3, *Chap. II.*), describe a circle cutting AA_1 at O and O_1. Then AO and A_1O are the *required lines.*

For, AM being the half sum, and MO the half difference, AO is (*Theor.* 1) the greater line, and A_1O or AO_1 the less line; and the two lines are therefore determined.

5. *Given the sum of two lines, and the rectangle under them, to find the lines.*

Bisect the given sum of lines, and on its half sum AB construct a square ABB_1A_1. Then



construct inwardly, on the side BB_1 of this square, a rectangle BOO_1B_1 (*Probs.* 2 *and* 3, *Chap. IV.*) equal to the given rectangle. The difference between the square AB_1 and this constructed rectangle, namely, the rectangle AO_1, is then (*Theor.* 9) equal to the square of the half difference of the required lines. Make, then, the construction (*Prob.* 3), taking A_1C_1, equal to

A_1O_1, and describing a semicircle on AC_1 for finding a square equal to the rectangle AO_1; and A_1D_1, the side of the square so found, will be the half difference of the lines. Their half sum and half difference being thus known, the lines are determined.

SEGMENTS.—*To divide a given line internally so that the rectangle under its segments be a given area.* This is the same problem as above; the segments required being two lines, the sum of which is given, and the rectangle under them also given.

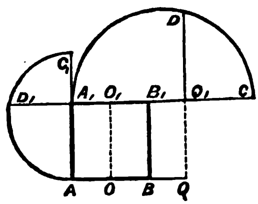
6. *Given the difference of two lines, and the rectangle under them, to find the lines.*

Using the same diagram as before, let AB be now the half difference of the lines, and ABB_1A_1 the square on AB. Construct, then, outwardly, on the side BB_1 of this square a rectangle BQQ_1B_1 equal to the given rectangle. The sum, then, of the square AB_1 and this constructed rectangle, namely, the rectangle BQ_1, is (*Theor.* 9) equal to the square of the half sum of the required lines. Make, then, the construction (*Prob.* 3), taking Q_1C equal to QQ_1, and describing a semicircle on A_1C, for finding a square equal to the rectangle AQ_1; and Q_1D, the side of the square so found, will be the half sum of the required lines. The half sum and half difference of the lines being thus known, the lines are determined.

SEGMENTS.—*To divide a line externally, so that the rectangle under its segments be a given area.* This is the same problem as above; the segments required being two lines, the difference of which is given, and the rectangle under them also given.

7. *Given the sum of the squares of two lines, and the difference of their squares, to find the lines.*

Let ABB_1A_1 be half the rectangle equal to the given sum of squares. Then construct on the side BB_1 of this rectangle, both outwardly and inwardly, rectangles OB_1 and QB_1 equal to half the given difference of squares. The sum AQ_1 of these rectangles is, then (*Theor.* 7), equal to the greater square; and their difference AO_1 is equal to the less square. Make, then, the constructions (*Prob.* 3) for converting these rectangles into squares, taking Q_1C equal to Q_1Q, and A_1C_1 equal to A_1O_1, and describing the two semicircles on A_1C and AC_1; and the lines DQ_1 and D_1A_1 are the two required lines.

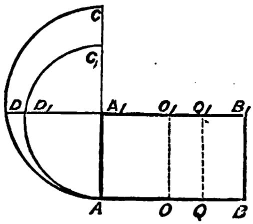


8. *Given the sum of the squares of two lines, and the rectangle under them, to find the lines.*

Using the same figure, let now the rectangle ABB_1A_1 be the given sum of squares of the required lines. Construct, then, both outwardly and inwardly on the side BB_1, two rectangles BQ_1 and BO_1 each equal to double the given rectangle. Then the rectangle AQ_1 is equal to the sum of the squares of the lines with two rectangles under them, and is therefore equal (*Theor.* 3) to the square of their sum. In like manner, the rectangle AO_1 is equal to the sum of the squares of the required lines less by double the rectangle under them, and is therefore (*Theor.* 5) equal to the square of their difference. Make, then, the constructions (*Prob.* 3) for converting these rectangles into squares, taking Q_1C equal to Q_1Q, and A_1C_1 equal to A_1O_1, and describing the two semicircles on A_1C and AC_1; and the lines DQ_1 and D_1A_1 are respectively the sum and difference of the required lines. The sum and difference of the lines being thus known, the lines are determined.

9. *Given the sum of two lines; and the sum of their squares, to find the lines.*

Bisect the rectangle given equal to the sum of squares of the required lines; and let ABB_1A_1 be the half rectangle. Construct, then, on the side B_1B inwardly a rectangle BO_1 equal to the square of the half given sum of lines. Then the rectangle AO_1 is (*Theor.* 8) equal to the square of their half difference. Make, then, the constructions (*Prob.* 3) for converting this rectangle into a square, taking A_1C_1 equal to A_1O_1, and describing a semicircle on AC_1; and the perpendicular D_1A_1 on AC_1 is the half difference of the lines. Their half sum and half difference being thus both known, the lines are determined.



SEGMENTS.—*To divide a given line internally, so that the sum of the squares of the segments be a given magnitude.* This is the same problem as the above in another form. The given line being required to be cut internally, the sum of its segments is that line. Therefore the segments are two lines, the sum and sum of squares of which are given.

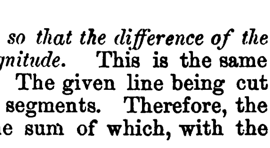
10. *Given the difference of two lines, and the sum of their squares, to find the lines.*

Using the same figure, bisect the rectangle given equal to the sum of squares of the required lines; and let ABB_1A_1 be the half rectangle. Construct on the side BB_1, *inwardly*, a rectangle QB_1 equal to the square of the half given difference of lines. Then the rectangle AQ_1 is (*Theor.* 8) equal to the square of their half sum. Make, then, the constructions (*Prob.* 3) for converting this rectangle into a square, taking A_1C equal to A_1Q_1, and describing a semicircle on AC; and the perpendicular DA_1 on AC is the half sum of the lines. Their half sum and half difference being thus both known, the lines are determined.

SEGMENTS.—*To divide a given line externally, so that the sum of the squares of the segments be a given magnitude.* This is the same problem as above in another form. The given line being cut externally, the difference of the segments is that line. Therefore, the segments required are two lines, the difference of which, with the sum of their squares, are given.

11. *Given the sum of two lines, and the difference of their squares, to find the lines.*

Let AB be the given sum of the required lines. Construct on it (*Probs.* 2 *and* 3, *Chap. IV.*) a rectangle, ABA_1B_1, equal in area to the rectangle given equal to the difference of the squares. The side BA_1 of this rectangle, adjacent to AB, is then (*Theor.* 6) the difference of the required lines. The sum AB, and the difference BA_1 of these lines being thus known, the lines themselves are determined.



SEGMENTS.—*To cut a line internally, so that the difference of the squares of the segments be a given magnitude.* This is the same problem as above in a different form. The given line being cut internally, that line is the sum of the segments. Therefore, the segments required are two lines, the sum of which, with the difference of their squares, are given.

12. *Given the difference of two lines, and the difference of their squares, to find the lines.*

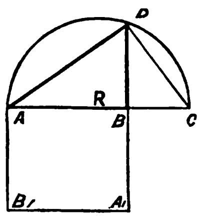
Using the same diagram, let CD be the given difference of the required lines. Construct on it (*Probs.* 2 *and* 3, *Chap. IV.*) a rectangle CDC_1D_1 equal in area to the rectangle given equal to the

difference of the squares. The side DC_1 of this rectangle, adjacent to CD, is then (*Theor.* 6) the sum of the required lines. The difference CD, and the sum, DC_1, of these lines being thus known, the lines themselves are determined.

SEGMENTS.—*To cut a line externally, so that the difference of the squares of the segments be a given magnitude.* This is the same problem as above in a different form. The given line being cut externally, the line is the difference of the segments. Therefore, the segments required are two lines, the difference of which, with the difference of their squares, are given.

13. *Given the difference of the squares of two lines, and the rectangle under them, to find the lines.*

As this problem cannot be solved by the same method as the former nine were, a special construction is required. Construct a square (*Prob.* 3) equal to the given difference of the squares of the required lines; and let AB be a side of this square. Construct, then, on AB a rectangle, ABA_1B_1, equal to the given rectangle under the lines. Cut AB, then, externally at C (*Prob.* 6, *Segments*), so that the rectangle under AC and BC may be equal to the square of A_1B; and on AC describe (*Prob.* 15, *Chap. II.*) a semicircle; and then produce A_1B to meet this semicircle at D. Join AD and CD; and the lines AD and DB are the two required.



For, since DB is perpendicular to AB, the difference of the squares of AD and BD is (*Theor.* 14) equal to the square of AB, and therefore (by construction) equal to the given difference of squares. Also, since the angle at D in the semicircle is a right angle, and DB perpendicular to AC, the rectangle under AC and BC is equal (*Theor.* 14, *Cor.* 1) to the square of DC. But this rectangle, by construction, is equal to the square of A_1B. Therefore DC is equal to A_1B; and the rectangle under AB and A_1B is equal to the rectangle under AB and DC. But (*Theor.* 25) the rectangle $AB \cdot DC$ is equal to rectangle $AD \cdot DB$, and therefore to the rectangle AA_1. Therefore the lines AD and BD, having the given difference of squares, and the area of the rectangle under them being that of the given rectangle, are the *lines required.*

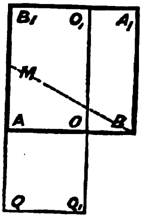
The ten problems being thus solved, it may be well to state

the solutions of the first nine in the convenient brevity of arithmetical language, keeping in mind that the terms are used *geometrically*, as already explained. The following numbers, from 4 to 12 inclusive, are the numbers of the problems.

4. Add the half difference of the lines to the half sum, and we have the greater line. Take the half difference from the half sum, and we have the less line.
5. From the square of the half sum take the given rectangle, and we have the square of the half difference of the lines, and therefore the difference itself. The half sum and half difference of the lines being thus known, the lines are determined.
6. To the square of the half difference of the lines add the given rectangle; and we have the square of the half sum, and therefore the sum itself. The half sum and half difference being thus known, the lines are determined.
7. Add half the difference of the squares of the lines to half the sum of their squares, and we have the greater square. Take half the difference of the squares of the lines from half the sum of their squares, and we have the less square. The squares being thus known, their sides, the required lines, are determined.
8. Add double the given rectangle to the given sum of squares, and we have the square of the sum of the two lines. Take double the rectangle from the sum of the squares, and we have the square of the difference of the lines. The squares of the sum and difference being thus known, their sides, the sum and difference of the lines themselves, are known, and the lines determined.
9. From half the given sum of squares take the square of half the given sum of lines, and we have the square of their half difference. The half sum and half difference of the lines being thus known, the lines are determined.
10. From half the given sum of squares take the square of half the given difference of the lines, and we have the square of their half sum. The half sum and half difference being thus known, the lines are determined.
11. Divide the given difference of squares of the lines by their sum, and the quotient is the difference of the lines. Their sum and difference being thus known, the lines are determined.
12. Divide the given difference of squares of the lines by the difference of the lines, and the quotient is the sum of the lines. The sum and difference of the lines being thus known, the lines are determined.

14. *To divide a given line AB so that the rectangle under the whole line and one segment be equal to the square of the other segment.*

On AB construct a square ABA_1B_1, and bisect the side AB_1 of this square at M. Join M, then, with B, and produce MA to Q, so that MQ be equal to MB. Then cut off AO from AB, equal to AQ; and AB will be cut at O, so that the rectangle under AB and BO be equal to the square of AO.



For, drawing through O and Q the lines O_1OQ and QQ_1 respectively, parallel to B_1Q and AB, and meeting in Q_1, since AO, by construction, is equal to AQ, the figure AQQ_1O is a square. Also, since AB_1 is bisected at M and cut *externally* at Q into segments B_1Q, AQ, the rectangle under B_1Q and AQ is equal (*Theor.* 11) to the square of MQ, less by the square of MA. But MQ, by construction, is equal to MB. Therefore the rectangle $B_1Q{\cdot}AQ$ is equal to the difference of the squares of MB and MA, that is, equal (*Theor.* 14) to the square of AB. Consequently, since QQ_1 is equal, by construction, to AQ, the rectangle B_1Q_1 is equal to the square on AB. Take away the rectangle AO_1 from both the square on AB and the rectangle B_1Q_1; and the square QO is equal to the rectangle OA_1. But, since A_1B is equal to AB, the rectangle OA_1 is equal to the rectangle under the whole line AB and its segment BO; and the square QO is the square of the other segment AO. The line AB is therefore divided, *as required.*

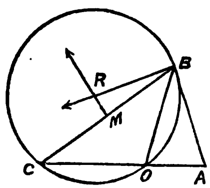
COR. 1. It is evident from the above construction that, in order to divide the given line as required, it was necessary to cut *externally* at Q another line AB_1, equal to AB, so that the rectangle $B_1Q{\cdot}AQ$ under its segments should be equal to a given magnitude, namely, the square of AB. This can be done by Problem 6; but, the given magnitude being the square of the very line to be cut externally, a special construction applies. The result is, that the greater external segment B_1Q is cut at A similarly to the given line AB itself at O; that is, so that the rectangle under the whole line B_1Q and one segment AQ be equal to the square of the other segment AB_1.

N.B.—Lines divided in this manner are generally described as

"cut in extreme and mean ratio;" because the greater segment is a mean proportional to the whole line and less segment. We shall, in future, describe it as such, merely for the sake of a name, the meaning of which will be understood in the Sixth Chapter.*

15. *Given one of the two equal sides, AC, of an isosceles triangle, to construct it so that each base angle be double the vertical angle.*

Divide the line AC at O (*last Prob.*) in extreme and mean ratio. Then construct on the less segment OA, as base (*Prob.* 1, *Chap. III.*), an isosceles triangle OBA, the two equal sides, OB, AB, of which shall be each equal to the greater segment CO. Join B with C. The triangle ACB is then the one required.



For, since AB is, by construction, equal to CO, the rectangle under CA and OA is equal to the square of AB. Therefore (*Theor.* 23) AB is a tangent to the circle R, which may (*Prob.* 5, *Chap. III.*) be described through C, O, and B. Hence the angle OBA is equal (*Theor.* 16, *Chap. II.*) to the angle BCO in the alternate segment BCO of the circle. But, since BO and CO are equal, the angles OCB and OBC are equal, and, consequently, the angle ABC is double ACB. Also, the angle AOB, being external, is double of ACB internal in the triangle BOC. But, the triangle OAB being isosceles, the angle OAB is equal to AOB, and therefore, also, double the angle ACB. The angles, therefore, at the base AB are each double the angle ACB at the vertex of the triangle ACB, *as required.*

COR. 1. Hence, the base of an isosceles triangle being given, the triangle may be constructed so that its base angles be each double the vertical angle. Let the base be produced, and cut it then externally so that the rectangle (*Cor.* 1, *last Prob.*) under the external segments be equal to the square of

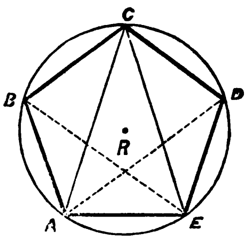
* It is desirable that a short expression should be found to denote a line cut in this peculiar way. It has been suggested to speak of the line as cut by "medial section," but it would be more expressive to describe it by its peculiar ratio, and name it a "line cut in medial ratio."

the base, and then the greater external segment will be the side of the required isosceles triangle. Its three sides being thus known, the triangle may (*Prob.* 1, *Chap. III.*) be constructed.

COR. 2. An isosceles triangle may also be inscribed in a circle, the base angles of which may be each double the vertical angle, by making, first, by the above problem, any triangle of that species, and then, by Problem 8, Chapter III., inscribing a triangle equiangular with it.

16. *To inscribe a regular pentagon in a given circle R.*

Construct (*last Prob.*) any isosceles triangle, the base angles of which shall be each double the vertical angle. Inscribe then (*Prob.* 8, *Chap. III.*), in the given circle R, a triangle ACE equiangular with the one so constructed; and bisect the base angles AEC and EAC by the lines EB and AD, meeting the circle in B and D. Join A with B, then B with C, C with D, and D with E; and the figure $ABCDE$ so formed is the required pentagon.



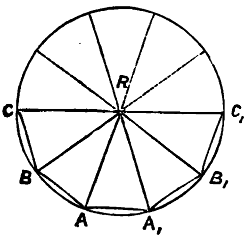
For, since the base angles AEC and EAC are bisected by EB and AD, the angles AEB, BEC, CAD, DAE, being each half the base angles of the triangle ACE, are each equal to ACE. But these five angles at the circumference standing on the arcs AB, BC, CD, DE, and EA being equal, the arcs themselves are equal; and therefore their chords, the sides of the pentagon, are equal.

The angles of the pentagon are also equal, being each evidently three times the vertical angle ACE, and therefore each six-fifths of a right angle.

COR. 1. Hence a regular quindecagon may be inscribed in a circle:—by first inscribing an equilateral triangle, and then inscribing a side of a regular pentagon, one extremity of which shall coincide with a vertex of the equilateral triangle. If the arc of the circle intercepted by these two sides be bisected, the chord of the half arc will, in magnitude, be the side of the regular quindecagon.

17. *To inscribe a regular decagon in a given circle R.*

On any radius RA of the given circle construct an isosceles triangle ARA_1, the base angles at A and A_1 of which shall be each double the vertical angle at R. The point A_1 is then on the circumference, since RA_1 is equal to RA. Draw, then, through the centre R of the circle a diameter CC_1 parallel to AA_1; and also the bisectors RB and RB_1 of the angles ARC and A_1RC_1 to meet the circle in B and B_1. Join A with B and B with C, also A_1 with B_1, and B_1 with C_1, and these four joining lines, together with AA_1, will be the five sides of the required decagon in the lower semicircle.



For, since the diameter CRC_1 is parallel to AA_1, by construction, the angles ARC and A_1RC_1 are equal respectively to A_1AR and AA_1R, and, therefore, each double ARA_1. The five angles CRB, BRA, ARA_1, A_1RB_1, and B_1RC_1 are therefore equal, and consequently the arcs and chords on which they stand are equal; that is, the five sides of the half decagon, CB, BA, AA_1, A_1B_1, and B_1C_1 are equal. The angles also of this half decagon at B, A, A_1, and B_1 are equal, being each double the base angle AA_1R of the triangle ARA_1, or to eight-fifths of a right angle.

This construction and proof applies only to the lower half of the circle; but it is evident that, on producing BR, AR, A_1R, and B_1R to meet the upper semicircle, the other five sides may be constructed.

Cor. 1. Hence it follows, from the construction, that the side of a regular decagon inscribed in a circle is equal to the greater segment of the radius cut in extreme and mean ratio.

EXERCISES.

1. Divide a line so that the rectangle under the whole line and one segment be an area of given magnitude.

2. If a line be drawn from a given point to a given directive, and be divided so that the rectangle under the whole line and its

segment adjacent to the given point be a given magnitude, the locus of the point of section will be a circle. Find the centre and radius of the circle.

3. Prove the same if, for the directive, a circle be substituted. Find also the centre and radius.

4. Draw from a given point a directive intersecting two given directives, so that the rectangle under the distances of the points of section from the given point be a given magnitude. There are two solutions.

5. Do the same, substituting for one of the directives a circle.

6. Draw a directive through any given point, cutting a given circle so the chord cut off from the directive by the circle be a given magnitude, and distinguish between the cases where the point is within and without the circle.

7. Draw a directive through any given point, cutting a given circle so that the sum of the squares of the distances of the points of section from the given point be a given magnitude. If the point be on the circle, what form does the problem take?

8. Describe a circle through two given points, touching a given directive.

9. Describe a circle through one given point, touching two given directives.

10. Describe a circle through two given points, touching a given circle. Show that there are two circles, one touching the given circle externally and the other internally.

11. Prove that the area of a triangle is that of a rectangle under its semiperimeter (or half sum of the three sides) and the radius of its inscribed circle.

12. Prove that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals. Verify this by the case of the square.

13. Prove that the sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of its diagonals, with four times the square of the line joining their points of bisection.

14. Prove that three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the bisectors of the sides drawn from the vertices of the triangle.

15. Prove that the sum of the squares of the four lines drawn from any point to the corners of a square is equal to four times the square of the line drawn to its centre, with one square of its diagonal.

16. The sum of the squares of the six lines drawn from any point to the six corners of a *regular* hexagon inscribed in a circle is equal to six times the sum of the squares of the radius and line joining the point with the centre of the circle.

17. Being given base and vertical angle of a triangle, construct it so that the sum of the squares of its sides be a given magnitude. When will the sum of the squares be the greatest possible?

18. Being given the base and area of a triangle, construct it so that the sum of the squares of the sides be a given magnitude. When will the sum of the squares be the least possible?

19. Being given base, sum of squares of sides, and difference of squares of sides, construct the triangle.

20. Draw from two given points two lines meeting on a given directive so that the sum of their squares be a given magnitude. When will the sum of the squares be the least possible?

21. Draw from two given points two lines meeting on a given circle so that the sum of their squares be a given magnitude. Determine the points on the circle for which the sum of the squares is the greatest and least possible.

22. Through two given points describe a circle which shall bisect the circumference of a given circle.

23. Through one given point describe a circle which shall bisect the circumferences of two given circles.

24. Given base and sum of sides of a triangle, the locus of the foot of the perpendicular from either extremity of the base on the bisector of the external vertical angle is a circle, the radius of which is half the given sum of sides, and centre the bisection of the base.

25. Prove that the rectangle under the perpendiculars from the extremities of the base (*last Exercise*) on the bisector of the external vertical angle is a constant magnitude, equal to the difference of the squares of the half sum of sides and half base.

26. Given base and difference of sides of a triangle, the locus of the foot of the perpendicular from either extremity of the base on the bisector of the internal vertical angle is a circle, the radius of which is half the given difference of sides, and centre the bisection of the base.

27. Prove that the rectangle under the perpendiculars from the extremities of the base (*last Exercise*) on the bisector of the internal vertical angle is a constant magnitude, equal to the difference of the squares of the half base and half difference of sides.

CHAPTER VI.

RATIO AND PROPORTION.

In the preceding chapters magnitudes were considered only in reference to absolute quantity or position, or as wholes divided into parts greater and less, or as bisected. In the case of bisection there is, indeed, a relation as to quantity implied, namely, the relation of equality; but beyond that no comparison of relations of magnitude has been made. In the present chapter the subject will be fully considered, commencing with an explanation of the terms Ratio and Proportion, which will be in constant use.

RATIO.

Ratio is Relative Magnitude; or, as Euclid defines it, "A mutual relation of two magnitudes of the same kind to one another, in respect of quantity."

The first of the two magnitudes in a ratio is termed the Antecedent, and the second the Consequent.

The Reciprocal of a Ratio is the ratio inverted, the second term of the ratio being made the antecedent and the first term the consequent.

When the terms of a ratio are said to be of the "same kind," it is meant that the less magnitude can be increased by addition or multiplication, so that it may be equal to or exceed the greater.

Thus two lines can have a ratio; for the less can be added to or multiplied so as to exceed the greater. If two lines are in magnitude, eleven and three; on the former being divided into eleven equal parts, the less contains only three of them; but by adding to the less eight parts, it becomes equal to the greater;

and by adding nine, exceeds it. So likewise, if the less be taken four times and the four lines be made into one, that line so made will exceed the greater line.

Similarly, arcs and angles, areas and solids, being each of a "separate kind," can each within its own kind have a ratio; but no arc can be increased so as to become equal to or exceed an area, nor an angle to exceed a cube, under any circumstances.

This relative magnitude, or ratio, may be either perceived as evident or ascertained by an arithmetic test.

Thus, if two directives diverge from a point and are crossed by any number of parallel directives, it seems quite self-evident that the triangular figures cut off, which are equiangular triangles, differ only in scale of magnitude, and that the corresponding pairs of sides have the same ratio. And this might well be made the basis of a truly *geometric* doctrine of ratio and proportion. But as the common usage, *in elementary geometry*, is to apply to ratio an arithmetic test, either by multiples or submultiples, we conform to the practice; but in the form of *submultiples*, as the simpler test.*

THE MEASURES OF A RATIO.—These are obtained by finding a magnitude termed a submultiple and ascertaining if it be contained without remainder in both antecedent and consequent, or by reducing the ratio to that of any equisubmultiples of both. They may be stated as follows:—

1st. *The Ratio of Two Magnitudes is the ratio of the Two Numbers which express how often any Common Submultiple of them is contained in each.*

2nd. *Two Magnitudes have to each other the Ratio of their Equisubmultiples.*

The latter measure is self-evident; the ratio of any two magnitudes being that of their fifth parts, or of

---

* *See* Appendix, Note 6.

their twelfth, or hundredth parts, which are all equi-submultiples of the two magnitudes. The former measure is, however, more useful; but it involves a difficulty from incommensurable quantities, as we shall see.

COMMENSURABLE QUANTITIES.—When two magnitudes have a common submultiple, this submultiple being contained in each magnitude a whole number of times for each, the ratio of the two magnitudes is that of the numbers expressing how often it is so contained. *For example,* two magnitudes, represented by the figures 684321 and 332763, have a unit in the last place of each a common submultiple; and, this unit being thus a common measure, the magnitudes are therefore commensurable; and the ratio of the two magnitudes is that of the two numbers expressing how often this unit is contained in both; that is, ·684321 to 332763. And this conclusion extends to any *decimal* quantities which have a terminable number of places; the unit in the lowest decimal place of that one which has the greater number of places is a submultiple of both magnitudes.

INCOMMENSURABLE QUANTITIES.*—When one of the magnitudes can be represented only by an interminable decimal, while the other is a finite whole number, or finite decimal, no finite common submultiple can exist; for, though a unit be selected in the last place of the whole number or finite decimal, yet the decimal represented by all the figures which follow the corresponding place in the interminable decimal, being less than that unit in that place and unknown in quantity, cannot be a common measure of the two magnitudes, and is only a remainder.

---

* *See* Appendix, Note 7.

If both magnitudes be represented by interminable decimals, there can be, equally as in the former case, no finite common submultiple. For, no matter how far down in the series of places a finite unit be taken, there are remainders less than that unit and unknown.

But, notwithstanding this difficulty, a unit can be found very much further down in the series of places which will answer the purpose of the common submultiple, and taken so infinitely small, that the magnitudes may be truly said to be as the numbers of times that this infinitesimal submultiple is contained in the two magnitudes.

PROPORTION.

PROPORTION is the Equality of Two Ratios. In a proportion there must, therefore, be always four terms: the first and third of which are the *antecedents*, and the second and fourth the *consequents*, and the proportion simply states that the ratio of the first to the second is equal to that of the third to the fourth.

When the second term is equal to the third, the proportion is said to be *continued*, the second term being continued, or repeated, in the third; and the proportion is thus made to consist apparently of only three terms when there are really four.

The middle term in this case is "the mean proportional," and the other two terms, "the extremes."

THE TESTS OF PROPORTION.—These are derived from the measures of ratio. If two ratios are equal, their measures must be the same. Hence:—

1st. *Four Magnitudes are Proportional when a submultiple of the first is contained in the second as often as the equisubmultiple of the third is contained in the fourth.*

2nd. *Four Magnitudes are Proportional when any equisubmultiples of the first and second are also equisubmultiples of the third and fourth.*

Hence may be deduced the following changes of proportion derivable from a given proportion :—

By Alternation.—When the first is to the second as the third to the fourth, the first is to the third as the second to the fourth.

By the first test of a proportion equisubmultiples of the first and third are contained an equal number of times in the second and fourth, and are therefore equisubmultiples also of the second and fourth. But, by the second measure of a ratio, the first is to the third in the ratio of their equisubmultiples, and the second to the fourth likewise in the ratio of their equisubmultiples. But the equisubmultiples for both ratios are the same. Therefore, by the second test of proportion, the alternation holds good—the first is to the third in the ratio of the second to the fourth.

By Inversion.—When the first is to the second as the third to the fourth, the second is to the first as the fourth to the third.

This is evident from the first test. For, since a submultiple of the first is contained in the second as often as an equisubmultiple of the third is contained in the fourth, it follows that a submultiple of the second is contained in the first as often as an equisubmultiple of the fourth is contained in the third, which *inverts* the ratio.

By Composition.—When the first is to the second as the third to the fourth, the sum of the first and second is to the first or second as the sum of the third and fourth is to the third or fourth.

For, since the submultiple of the first is contained in the second, it is contained in the sum of the first and second. So, likewise, the equisubmultiple of the third, being contained in the fourth, is contained in the sum of the third and fourth. Therefore, the sum of the first and second is to either first or second as the sum of the third and fourth is to either third or fourth.

By Division.—When the first is to the second as the third is to the fourth, the difference of the first and second is to the first or second as the difference of the third and fourth is to the third or fourth.

This is proved as the last was, the submultiples being contained in the differences as they were in the sums.

By Composition and Division.—When the first is to the second as the third to the fourth, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

This is a consequence of the preceding changes in the proportions, the submultiples being contained in the sums and differences as they were in the ratios of the original proportion.

Similar Figures.—Similar Rectilineal Figures are figures the angles of which are all respectively equal, and the sides about the equal angles proportional.

For example, two equiangular triangles are similar, for it has been proved (*Theor.* 3) that their sides about the equal angles are proportional. Also, all squares are similar to each other, the angles at their corners not only being equal, but their sides about the equal angles proportional in the ratio of equality. The definition guards the student against the possible mistake of supposing that all rectilineal figures which have equal angles are similar.

AXIOMS ON RATIO.

1. Equisubmultiples of the same or of equal magnitudes are equal to each other.
2. Magnitudes, the equisubmultiples of which are equal, are equal to each other.
3. Two magnitudes are to each other in the ratio of their equisubmultiples.

4. Two magnitudes are to each other in the ratio of the numbers of times that a common submultiple of them is contained in both.

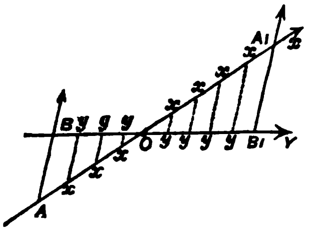
5. Magnitudes which have the same ratio to the same magnitude, or to equal magnitudes, are equal to each other.

SIMPLE RATIO.

We proceed now to apply these principles to intersecting directives and the triangles they form.

1. *The Segments of Two Divergent Directives cut off from their point of divergence by Two Parallel Directives are proportional.*

Let the divergent directives be x, Y, or AOA_1 and BOB_1, and AB, A_1B_1, the parallel directives. Then, $A\,x$ being a common submultiple of OA and OA_1 taken at intervals marked x on AA_1, let the lines xy be parallels to AB and A_1B_1. Then (*Theor.* 23, *Chap. III.*) the distances marked by the letter y on BB_1 are equal, and are the same submultiples of OB and OB_1 that Ax is of OA and OA_1. A submultiple, therefore, Ax of OA, is contained in OA_1 as often as the equi-submultiple By of OB is contained in OB_1. The segments OA, OA_1, and OB, OB_1 (*Test I. of Proportion*) are therefore proportional.



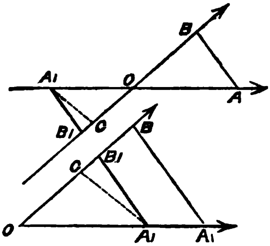
Hence (*by alternation*) it follows that OA is to OB as OA_1 is to OB_1.

Also (*by composition*), that AA_1 is to OA as BB_1 is to OB; or AA_1 to BB_1 as OA to OB.

2. *If two Divergent Directives intersect two Other Directives so that their Segments cut off from their point*

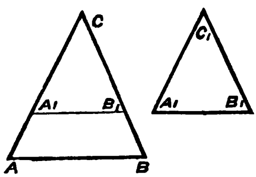
of divergence by the other directives be Proportional, these directives are parallel.

Let AOA_1 and BOB_1 (in either figure of the diagram) be the divergent directives, and AB, A_1B_1 those which they intersect, making OA to OA_1 as OB to OB_1. Then, if A_1B_1 be not parallel to AB, let A_1C be parallel. Then, since A_1C is parallel to AB, the ratio of OB to OC is (*Theor.* 1) equal to that of OA to OA_1. But, by supposition, the ratio of OB to OB_1 is also equal to that of OA to OA_1. Therefore OC is to OB as OB_1 to the same OB. Consequently, OC must be equal to OB_1, a part equal to the whole, which (*Ax.* 10) is impossible. The lines AB and A_1B_1 are therefore parallel.



3. *Equiangular Triangles have their pairs of Sides about the Equal Angles Proportional.*

Let ABC and $A_1B_1C_1$ be the two equiangular triangles; and suppose on the sides CA, CB, of the first triangle distances, CA_1, CB_1, taken equal to the sides C_1A_1 and C_1B_1 of the second triangle. Then, since the angles C and C_1 are equal in the two triangles A_1CB_1 and $A_1C_1B_1$, and the sides CA_1 and CB_1 are equal respectively to C_1A_1 and C_1B_1, the angles at A_1 and B_1 in the two triangles (*Theor.* 9, *Chap. III.*) are equal. But the angles at A and B in the first triangle are, by supposition, equal to A_1 and B_1 in the second, and consequently they are equal to A_1 and B_1 in the triangle A_1CB_1. Therefore (*Theor.* 10, *Chap. I.*), the line A_1B_1 in the triangle ACB is parallel to AB; and (*Theor.* 1) AC is to A_1C as BC is to B_1C, that is, AC is to A_1C_1 as BC is to B_1C_1; and (*by alternation*) AC is to BC as A_1C_1 is to B_1C_1. Therefore, the sides about the equal angles C and C_1 are proportional, *as stated.*

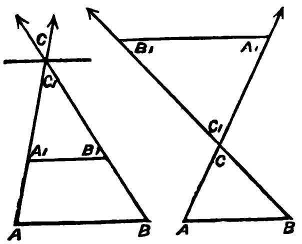


In like manner, it may be proved that the sides about the other pairs of equal angles A, A_1, and B, B_1 in the two triangles are proportional; and that the triangles are similar.

4. *If Two Triangles have an Angle Equal in each,*

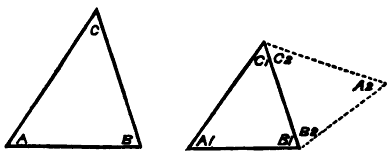
and the Sides about the equal angles Proportional, the triangles are equiangular, and have their other sides about their other equal angles proportional.

Let ABC, $A_1B_1C_1$ be the two triangles, C and C_1 the equal angles, and AC and BC, A_1C_1, and B_1C_1 the proportional sides about the equal angles C and C_1. Then, if the sides AC, A_1C_1, be placed together *in directum*, since the angles C and C_1 are equal, the sides BC and B_1C_1 will be also *in directum*. Hence two directives, AA_1 and BB_1, are cut by two other directives, AB, A_1B_1, making the segments AC, A_1C_1, and BC, B_1C_1 proportional. Therefore (*Theor.* 2) the sides AB and A_1B_1 are parallel; and the triangles are equiangular, and also (*Theor.* 3) have their other sides proportional, *as stated.*



5. *If Two Triangles have their Three pairs of Sides respectively Proportional, they are equiangular.*

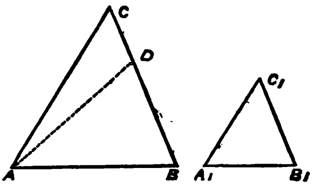
Let ABC, $A_1B_1C_1$ be the two triangles, the sides AB, BC, CA being proportional to A_1B_1, B_1C_1, C_1A_1; and, on the side B_1C_1 of the second triangle, suppose a triangle constructed having the angles B_2, C_2 equal to the angles B, C, of the first triangle. The remaining angle A_2 is then evidently equal to the angle A. The triangles ABC and $A_2B_2C_2$ are therefore equiangular; and AC is to BC as A_2C_2 is to B_2C_2. But AC is also, by supposition, to BC as A_1C_1 to B_1C_1. Therefore A_2C_2 is to B_1C_1 as A_1C_1 is to B_1C_1, and consequently A_2C_2 is equal to A_1C_1. In like manner, it may be proved that A_2B_2 is equal to A_1B_1. The two triangles $A_1B_1C_1$, $A_2B_2C_2$, therefore (*Theor.* 8, *Chap. III.*), have their angles equal. Therefore, the angles of the triangles ABC and $A_1B_1C_1$ are equal; and they are equiangular triangles, *as stated.*



6. *If Two Triangles have Two Sides in one Proportional to two sides in the other, and the Angles Opposite*

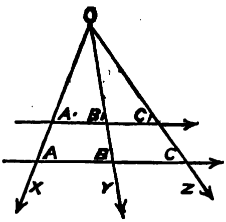
One Pair of the homologous sides Equal, the angles opposite the other pair of homologous sides will be also equal, and the triangles similar, if the angles opposite the remaining pair of sides be either Both Obtuse or Both Acute.

Let ABC and $A_1B_1C_1$ be the two triangles; C and C_1 the two equal angles; and AC and AB the sides proportional to A_1C_1 and A_1B_1. Then, if the angles A and B be not equal to A_1 and B_1, let the angle CAD in the first triangle be equal to A_1; and since, also, C is equal to C_1, the triangle CAD (*Theor.* 3) must be similar to $C_1A_1B_1$, and CA be to AD as C_1A_1 is to A_1B_1. But CA is, by supposition, to AB as C_1A_1 to A_1B_1. Therefore AD is equal to AB, and the triangle ABD is isosceles; and consequently the external angle ADC must be obtuse. Hence the angle B_1, proved equal to ADC, must also be obtuse. But B_1 and B are both acute angles or both obtuse, by supposition. If they be both acute; B_1, an acute angle, is equal to ADC, an obtuse angle, which (*Ax.* 3) is impossible. If they be both obtuse; then B, an obtuse angle, is a base angle of the isosceles triangle ABD, which is also impossible (*Theor.* 14, *Cor.* 5, *Chap. I.*), unless the points B and D coincide. The two triangles, consequently, must be similar, and their remaining pair of sides, BC and B_1C_1 (*Theor.* 3), be proportional to the other pairs of sides, *as stated.*



7. *The Segments cut off from Two Parallel Directives by Three Directives Diverging from a Point are proportional.*

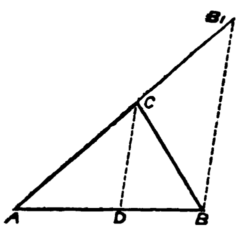
Let X, Y, and Z be the directives diverging from O; and AB, BC, and A_1B_1, B_1C_1, the segments cut off from the parallel directives AC, A_1C_1. Then, since the triangles AOB and A_1OB_1 are equiangular, AB (*Theor.* 3) is to A_1B_1 as OB is to OB_1. In like manner, in the equiangular triangles, BOC and B_1OC_1, BC is to B_1C_1 as OB is to OB_1. Therefore, the ratio of AB to A_1B_1 is equal to that of BC to B_1C_1. And (*by alternation*) AB is to BC as A_1B_1 to B_1C_1, *as stated.*



CONVERSE.—Hence it follows that, if two parallel lines AC and A_1C_1 (as in the above Figure) be divided at B and B_1 in the same ratio, the directive Y through the points of section, B and B_1, must pass through the point of intersection O of the directives X and Z through the extremities of the lines. For, if Y do not pass through O, the line from O through B_1 will cut AC, as above proved, in the ratio of A_1B_1 to B_1C_1. But this ratio is equal to that of AB to BC; and AC is thus cut twice in the same ratio, which is impossible, since the segments of AC in both sections are measured similarly from its extremities. The directive Y, therefore, passes through the intersection of X and Z, *as stated.*

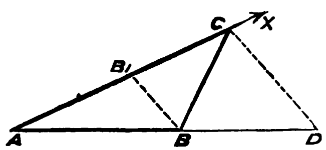
8. *The Bisector of the Vertical Angle of a Triangle cuts the base in the ratio of the sides.*

Let ACB be the triangle, and CD the bisector of the vertical angle C, meeting the base AB in D. Let, also, BB_1 be a line through B parallel to CD, meeting AC produced in B_1. Then, since BB_1 is parallel to CD, the angles ACD and BCD are equal respectively to CB_1B and CBB_1. And therefore, since the angle ACB is bisected by CD, the angles CBB_1 and CB_1B are equal, and the triangle CBB_1 is isosceles, and CB_1 is (*Theor.* 6, *Chap. III., Converse*) equal to CB. The ratio, therefore, of AC to BC is that of AC to CB_1. But (*Theor.* 1) AC is to CB_1 as AD to BD; and, consequently, AC is to BC as AD to BD, *as stated.*



9. *The Bisector of the External Vertical Angle of a Triangle cuts the base externally in the ratio of the sides.*

The proof is nearly identical with the last. Let ACB be the triangle, and CD the external bisector of the vertical angle meeting the base externally in D. Let, also, BB_1 be a line through B, parallel to CD, meeting AC in B_1. Then, since BB_1 is parallel to CD, the angles XCD and BCD are equal (*Theor.* 8 *and* 9, *Chap. I.*) to CB_1B and CBB_1; and therefore,



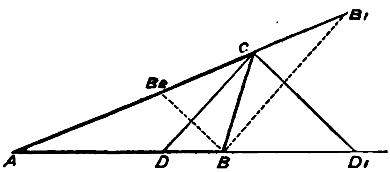
since also the external vertical angle XCB is bisected by CD, the angles CBB_1 and CB_1B are equal, and CB_1 is (*Theor.* 6, *Chap. III., Converse*) equal to CB. The ratio, therefore, of AC to BC is that of AC to B_1C. But (*Theor.* 1) AC is to CB_1 as AD to BD; and, consequently, AC is to BC as AD to BD, *as stated.*

COR. 1. From this and the preceding analogous theorem, it follows that the internal and external bisectors of the vertical angle of a triangle cut the base internally and externally into proportional segments, in the ratio of the sides.

COR. 2. It is hence also evident that, if the base AB and the ratio of the sides AC, BC, of a triangle ACB be given, the *locus* of its vertex is a circle described on the distance from each other, as diameter, of the two points in which AB is cut internally and externally in the given ratio.

10. *If Two Lines drawn, one internally, the other externally, from the Vertical Angle of a triangle cut the Base into internal and external Segments Proportional to the Sides, these lines contain a right angle.*

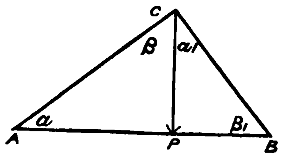
Let ACB be the triangle, and CD and CD_1 the two lines cutting the base AB at D and D_1 into segments, so that the ratios of AD to BD and AD_1 to BD_1 be equal to those of the sides. Suppose, then, CB_1 and CB_2 to be taken equal to CB, and B joined with B_1 and B_2. Then, since



CB, CB_1 and CB_2 are equal to each other, a circle with C as centre can be made to pass through B, B_1, and B_2, of which B_1B_2 should be a diameter. Hence (*Theor.* 12, *Chap. II.*) B_1BB_2 is a right angle. Moreover, since AD is (by supposition) to BD as AC to BC, it is to BD as AC to B_1C. Therefore (*Theor.* 2.) CD is parallel to BB_1. In like manner, since AD_1 is to BD_1 as AC to BC, it is to BD_1 as AC to CB_2. Therefore CD_1 is parallel to BB_2. But the angle between BB_1 and BB_2 has been proved to be a right angle. Therefore the angle DCD_1 between the lines parallel to them, CD and CD_1, is also a right angle, *as stated.*

11. *The Perpendicular from the Vertex of a Right-angled Triangle on the Hypothenuse divides the triangle into two triangles, which are similar to the whole and to each other.*

Let ACB be the right-angled triangle, and CP the perpendicular from C on the hypothenuse AB, and α, β, α_1, β_1 the acute angles of the triangles APC and BPC. Then, in the two triangles ACB and APC, the angle α is common to both, and their angles at C and P are right angles. The angles β and β_1 are therefore (*Theor.* 14, *Cor.* 4, *Chap. I.*) equal, and the two triangles equiangular, and therefore similar. In like manner, it may be proved that the triangle BCP is similar to ACB. Hence the triangles APC and BPC, being similar to the same ACB, are similar to each other, *as stated.*

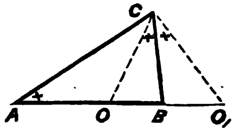


Cor. 1. Hence it is evident, since APC and BPC are similar, that AP opposite β is to PC opposite α, in the triangle APC, as PC opposite β_1 is to BP opposite α_1 in the triangle BPC; that is, the perpendicular CP is a mean proportional to the segments, AP and BP, of the hypothenuse.

The following theorem is an extension of that which has been just proved :—

12. *Any Line drawn from the Vertex of a Triangle to the Base, making with either of its sides an angle equal to the angle opposite that side, forms, with that side and the adjacent segment of the base, a triangle similar to the given triangle.*

Let ABC be the triangle, and CO the line drawn from the vertex C to the base AB, so that the angle BCO be equal to BAC. Then, since the angle ABC is common to the two triangles ABC and OBC, and the angle BCO, by supposition, equal to BAC, the remaining angles ACB and BOC (*Theor.* 14, *Chap. I.*) are equal; and the triangles ABC and COB are equiangular, and therefore similar, *as stated.*

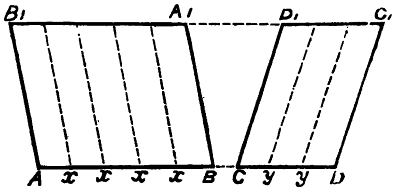


COR. 1. It hence follows that the side BC is a mean proportional to the side AB and the segment BO; and also that, if the angle ACB become a right angle, the angle BOC must become right, and the theorem last proved be reproduced. The two right-angled triangles into which OC would divide the triangle would then be similar to the whole and to each other.

N.B.—If the line from the vertex cuts the base externally at O_1, making, with the side AC of the triangle, an angle ACO_1 equal to the external angle CBO_1 at B, and opposite to AC, a similar conclusion holds good—the triangles ACO_1 and BCO_1 are equiangular and similar.

13. *Parallelograms on different bases, but between the same parallels, are proportional to their bases.*

Let ABA_1B_1 and CDC_1D_1 be the two parallelograms on the bases AB, CD, and between the parallels B_1C_1 and AD. Then, let Ax be a common submultiple of AB and CD, and suppose it taken at the intervals marked x on AB and the intervals marked y on CD. Then, since the distances Ax, Cy are all equal, the parallelograms into which the parallelograms AA_1 and CC_1 are divided, are all equal, being on equal bases and between the same parallels. Therefore, the parallelogram on Ax is the same submultiple of the parallelogram on AB as Ax is of the base AB. And, similarly, the equal parallelogram on Cy is the same submultiple of the parallelogram on CD as Cy is of CD. Therefore the submultiple parallelogram on Ax of parallelogram AA_1 is contained in the parallelogram CC_1 as often as the equisubmultiple Ax of AB is contained in CD; and therefore the parallelograms AA_1 and CC_1 are proportional to their bases AB and CD, *as stated.*



COR. 1. If the parallelograms be rectangles between the same parallels, they have each a pair of equal parallel sides perpendicular to their bases; and therefore the rectangles are, in the ratio of their other pair of sides, the bases; and if the rectangles

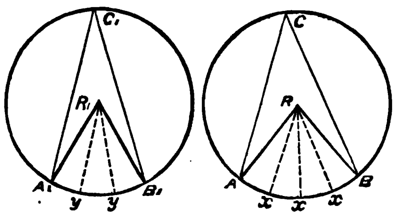
be on equal bases, but between different parallels, they are to each other as the distances between the parallels; that is, as their altitudes.

COR. 2. As the areas of triangles are the halves of those of parallelograms, it follows that triangles on different bases and between the same parallels are proportional to their bases.

COR. 3. Also, as the areas of triangles are the halves of rectangles under their bases and perpendiculars, they are to each other as their bases if their perpendiculars are equal, and as their perpendiculars if their bases are equal.

14. *In the Same Circle, or in Equal Circles, Angles subtended at the centre are to each other in the Ratio of the Arcs on which they stand.*

Let the arcs of the two equal circles be AB and A_1B_1, and ARB and $A_1R_1B_1$ the angles at the centres; and let Ax be a submultiple of AB, which is contained in A_1B_1 without any remainder. Then Ax is a common submultiple of AB and A_1B_1. Take it on AB and A_1B_1 as often as it will go. Then AB is divided at intervals, marked x, into a number of equal parts; and A_1B_1 is also so divided at intervals, marked y. The submultiple arcs being all equal, the corresponding angles at the centres, R and R_1, are also all equal; and their number in each circle is equal to that of the submultiple arcs in each. The submultiple angle ARx is therefore contained in $A_1R_1B_1$ as often as the equisubmultiple arc Ax is contained in A_1B_1, that is, a submultiple ARx of the first ARB is contained in the second $A_1R_1B_1$ as often as the equisubmultiple Ax of the third AB is contained in the fourth A_1B_1. Therefore the four magnitudes are proportional, *as stated.*

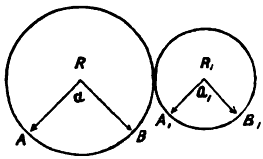


COR. 1. It is further evident, since the ratio of two magnitudes is the same as that of their halves, that the angles ACB and $A_1C_1B_1$ at the circumferences are proportional to the arcs AB and A_1B_1 on which they stand.

N.B.—The theorem has been proved for two equal circles, but it equally holds good for arcs and angles in the same circle.

15. *In Unequal Circles Arcs subtending Equal Angles at their Centres are proportional to the circumferences of the circles.*

Let R and R_1 be the two circles, and AB, A_1B_1 two arcs subtending at R and R_1 equal angles a and a_1. Then, the arc AB in the circle R is to the whole circumference of R (*last Theorem*) as the angle a is to four right angles. In like manner, in the circle R_1, the arc A_1B_1 is to the circumference of the circle R_1 as the angle a_1 is to four right angles. But angles a and a_1 are, by supposition, equal. Therefore, the arc AB is to the circumference of R as the arc A_1B_1 is to that of R_1; and, consequently (*by alternation*), AB is to A_1B_1 as the circumference of R is to that of R_1, *as stated.**



16. *If Two Triangles stand on Opposite Sides of a Common Base, the line joining the vertices of the triangles is cut by the base in the ratio of their areas.*

Let AB be the common base, and ACB and AC_1B the two triangles; also, O the point in which CC_1 cuts the base either internally or externally, and CP, C_1P_1 the perpendiculars from C and C_1 on AB. Then, since the triangles ACB and AC_1B are on the common base AB, they are (*Theor.* 13, *Cor.* 3) to each other as their altitudes CP and C_1P_1. But, in the two equiangular right-angled triangles, COP and C_1OP_1, CP is to C_1P_1 as CO is to C_1O. Therefore, the areas ACB and AC_1B are to each other as CO to C_1O, *as stated.*

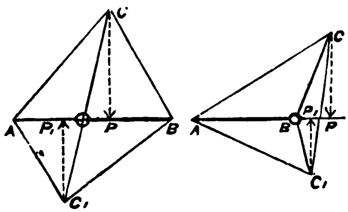


Fig. 1. Fig. 2.

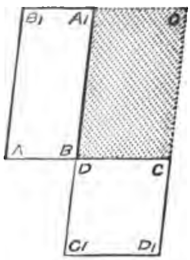
* This ratio of the circumferences is known to be equal to that of the radii of the circles; but the *elementary* geometry, to which we are confined, cannot prove it.

The second Figure illustrates the demonstration for the case where CC_1 cuts AB externally.

N.B.—When the two triangles are on the same side of the base, the line CC_1 becomes a line cut at O externally by AB, but still cut in the ratio of the areas of the triangles.

17. *Equiangular Parallelograms Equal in Area have their sides reciprocally proportional.*

Reciprocal ratio has been defined in the introductory remarks to this chapter; but in the present theorem its application may be stated thus:—When two equiangular parallelograms have their areas equal, *any side in one is to a side in the other as the remaining side in that other is to the remaining side in the first.* The inversion of the ratio takes place in the two last terms.

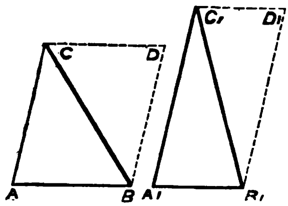


Let the equiangular parallelograms be AA_1 and CC_1, and let them be so placed that a pair of sides, AB, CD, be *in directum*. Then, since the angles B and D are equal, the other pair of sides A_1B and C_1D are also *in directum*; and, on producing D_1C and B_1A_1 to meet at O, the parallelogram BO is equiangular with AA_1 and CC_1. But, since AA_1 and CC_1 are equal in area, they have each the same ratio to the same parallelogram DO. But AA_1 is to DO (*Theor.* 13) as AB to DC; and also CC_1 is to DO as C_1D is to A_1B. Therefore, the ratio of AB to DC is equal to that of C_1D to A_1B; that is, a side AB in one is to a side DC in the other as the remaining side C_1D in the other is to A_1B, the remaining side in the first, *as stated.*

Cor. 1. *The converse* of this theorem holds good. For, if it be stated that AB is to CD as C_1D to A_1B; then, since the parallelograms AA_1 and CC_1 have these equal ratios to the same parallelogram DO, the parallelograms AA_1 and CC_1 (*Ax.* 5, *Chap. VI.*) must be equal in area, *as stated.*

18. *Two Triangles which have an Angle in each Equal, and their Areas also Equal, have their Sides about the equal angles reciprocally proportional.*

Let the triangles be ACB, $A_1C_1B_1$, having the angles at A and A_1 equal, and their areas equal. Then, suppose BD and CD to be parallels to AC and AB; and also B_1D_1 and C_1D_1 parallels to A_1C_1 and A_1B_1. Then, the figures AD and A_1D_1 are parallelograms double each in area to the triangles ACB, $A_1C_1B_1$; and therefore equal in area to each other and equiangular.



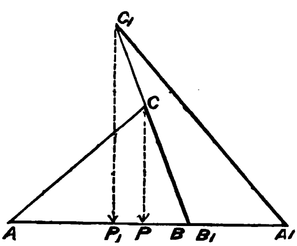
Therefore (*by last theorem*), their sides are reciprocally proportional. But their sides are the sides AB, AC and A_1B_1 and A_1C_1 of the triangles. Therefore the sides of the triangles are reciprocally proportional, *as stated.*

COR. 1. The *converse* of the theorem holds good. For, by doubling the triangles, as above, since the sides of the triangles are reciprocally proportional, those of the parallelograms are equally so, and their areas are equal; and, therefore, the areas of the triangles are equal.

COR. 2. Hence also it follows, by supposing the equal angles of the triangles to be each a right angle, that, if two right-angled triangles have their bases and altitudes reciprocally proportional, their areas are equal; and, *vice versâ*, that if their areas are equal, their bases and altitudes are reciprocally proportional.

19. *If Two Triangles have two pairs of Sides Reciprocally Proportional, and the Angles contained by them Supplementary to each other, their areas are equal.*

Let ABC and $A_1B_1C_1$ be the triangles; and suppose them so placed that the sides AB and A_1B_1 may be *in directum*, and the triangles be on the same side (the upper side) of AA_1. Then, since the angles B and B_1 are supplementary to each other, the side BC of the triangle ABC must fall on, and in its length coincide with, B_1C_1. Let, also, CP and C_1P_1 be perpendiculars from C and C_1 on AA_1. Then,

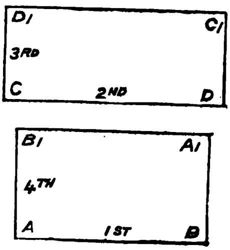


by the statement, AB is to A_1B_1 as B_1C_1 is to BC. But, since

CP and C_1P_1 are parallel to each other, B_1C_1 is to BC as C_1P_1 is to CP; and, therefore, AB is to A_1B_1 as C_1P_1 is to CP. Therefore, AB, A_1B_1, CP, and C_1P_1 are reciprocally proportional; and the rectangles under AB and CP, and A_1B_1 and C_1P_1, are (*Theor.* 17, *Cor.* 1) equal; and, consequently, the halves of the rectangles, the triangles ABC and $A_1B_1C_1$, are equal, *as stated.*

COR. 1. The *converse* of this theorem is evidently true, viz., that, if the areas of two triangles are equal, and a pair of angles, one taken from each, are supplementary, the sides about these angles are reciprocally proportional.

20. *If Four Right Lines be Proportional, the rectangle under the Extremes is equal to the rectangle under the Means.*



Let AB, CD, CD_1 and AB_1 be the four lines in the order of their proportion; and let also the rectangles AA_1 and CC_1 be respectively those under the extremes and the means. Then, since these rectangles are equiangular parallelograms, and have their sides reciprocally proportional, AB to CD as CD_1 to AB_1, their areas (*Theor.* 17, *Cor.* 1) are equal, *as stated.*

COR. 1. The *converse* of this theorem holds good, that if two rectangles have equal areas, their sides (*Theor.* 17) form a reciprocal proportion.

COR. 2. If the second and third terms become equal, the proportion is *continued;* and the first is to the second as the third is to the fourth; and, therefore, the square of the mean is equal to the rectangle under the extremes.

COR. 3. Hence, since the perpendicular from the vertex of a right-angled triangle on the hypothenuse is (*Theor.* 11) a mean proportional to its segments, the square of the perpendicular is equal to the rectangle under the segments, as has been already proved (*Theor.* 14, *Cor.* 2, *Chap. V.*).

COR. 4. Hence also, since either side of a right-angled triangle is a mean proportional to the hypothenuse and its segment,

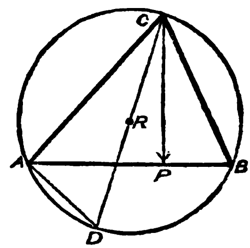
adjacent to that side made by the perpendicular, its square is equal to the rectangle under the hypothenuse and that segment.

COR. 5. In like manner, it may be shown that, in a triangle divided as in Theorem 12, the square of the side BC is equal to the rectangle under AB and BO; and the square of the line CO_1 is equal to the rectangle under AO_1 and BO_1.

COR. 6. It may also be proved that the rectangle under two lines is a mean proportional to their squares. For, the square of the greater line, since it and the rectangle under the extremes have a common side, namely, the greater line, is to the rectangle under the lines as the greater line is to the less. Also, the rectangle under the lines is to the square of the less line, since that line is a common side, as the greater line is to the less. These two ratios being thus equal, the statement holds good.

21. *The Rectangle under the Sides of a Triangle is equal to the rectangle under the Diameter of its circumscribing Circle and the Perpendicular from the vertical angle on the base.*

Let ABC be the triangle, and R its circumscribing circle, and CD the diameter through the centre R, and CP the perpendicular from the vertex C on the base AB. Suppose AD joined. Then, in the triangles ACD and BCP, the angle at A (CD being a diameter) is a right angle, and the angle at P is, by supposition, also right. The angles ADC and ABC, standing on the arc AC, are (*Theor.* 12, *Chap. II.*) also equal; and the triangles ACD and BCP are therefore equiangular and similar. Hence, AC opposite D is to CD opposite A as CP opposite B is to BC opposite P. Therefore, the rectangle under AC and BC, the extremes (*Theor.* 20), is equal to the rectangle under the means,

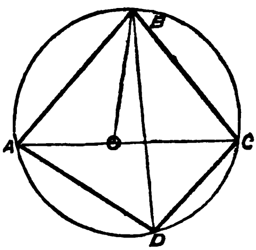


CD and *CP*; that is, the rectangle under the sides is equal to the rectangle under the diameter and perpendicular, *as stated.*

COR. 1. Hence it follows that the rectangles under the pairs of adjoining sides all round a triangle are to each other as the perpendiculars on the remaining sides from their opposite angles.

22. PTOLEMY'S THEOREM.—*The Sum of the Rectangles under the Opposite Sides of a Quadrilateral Inscribed in a Circle is equal to the rectangle under the diagonals.*

Let *ABCD* be the quadrilateral inscribed, and *AC*, *BD* its diagonals. Let, also, *BO* be a line making with *AB* an angle, *ABO*, equal to *CBD*. Then, since the angle *ABO* is (by supposition) equal to *CBD*, and the angles *BAO* and *BDC* equal, both standing on the arc *BC*, the triangles *ABO* and *CBD* are equiangular and similar; and *AB* is to *AO* as *BD* is to *DC*. The rectangle under *AB* and *DC* is therefore (*Theor.* 20) equal to the rectangle under *AO* and *BD*.



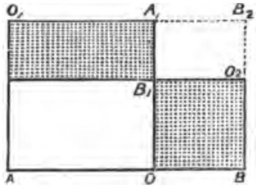
In like manner, adding the angle *OBD* to *ABO* and *CBD*, the angles *ABD* and *OBC* are equal; and also *BCO* and *ADB*, both standing on the arc *AB*, equal. Therefore the triangles *BOC* and *BDA* are similar; and *BC* is to *OC* as *BD* to *AD*; and consequently the rectangles under *BC* and *AD* and *BD* and *OC* are equal.

The sum, therefore, of the rectangles $AB \cdot CD$ and $BC \cdot AD$ is equal to the sum of the rectangles $BD \cdot AO$ and $BD \cdot CO$, or to the single rectangle under *BD* and the sum of *AO* and *CO*; that is, to the rectangle under the diagonals *BD* and *AC*, *as stated.*

COR. 1. If one of the diagonals *AC* becomes a diameter, the two triangles *ABC* and *ADC* become right-angled. Hence, the sum of the rectangles under the opposite sides of two right-angled triangles, on opposite sides of a common hypothenuse, is equal to the rectangle under the hypothenuse and the line joining the vertices of the right angles.

23. *The Greater Segment of a Line divided in Extreme and Mean Ratio being also Cut in the Same Ratio, its greater segment is equal to the less segment of the first divided line.*

Let AB be the line supposed first divided at O, so that the rectangle under AB and BO be equal to the square of AO, or that the whole line AB be to the greater segment AO as AO is to the less segment BO. On the segments AO and BO, suppose squares AOA_1O_1, and BOB_1O_2 constructed, and the upper side O_2B_1 of the less square BB_1 produced across the square AA_1. Then, since BO_2 is equal to BO, and the rectangle under AB and BO is equal to the square AA_1 on AO, the rectangle AO_2 under AB and BO_2 is equal to the square AA_1. Take from both the unshaded rectangle AB_1, and the remainders, the rectangle B_1O_1 and the square BB_1, are equal. But the side A_1O_1 of the square AA_1 is equal to A_1O; and therefore the rectangle $OA_1 \cdot A_1B_1$ is equal to the square of OB_1, that is, the line OA_1 is cut at B_1 in extreme and mean ratio. But OA_1 is equal to the greater segment AO of AB. Therefore, OB_1 is equal to the greater segment of AO so cut, and is equal to the less segment BO of the line AB, *as stated.*



COR. 1. Hence, it follows that, if the greater segment of AO, namely, OB, were cut in extreme and mean ratio, its greater segment would be equal to the less segment A_1B_1 of AO; and so on, if the division were continued in succession on all the greater segments, a series of greater segments in a continued proportion would be obtained, each greater term in any place in the series being equal to the less segment of the preceding term. Moreover, each less term becoming, in the next step, a greater segment, it is evident that the series of less terms is also in continued proportion.

COR. 2. The difference of the squares of the segments of a line cut in extreme and mean ratio is equal to the rectangle under them. For, since the rectangle B_1O_1 is equal to the square BB_1, the difference of the squares AA_1 and BB_1 is

equal to the difference of the square AA_1 and the rectangle B_1O_1; that is, to the unshaded rectangle AB_1, which is, evidently, the rectangle under the segments AO and BO.

COR. 3. Also, the rectangle under the whole line and the difference of the segments is equal to the rectangle under the segments. This follows from the preceding corollary. The difference of the squares of the segments is (*Theor.* 6, *Chap. V.*) equal to the rectangle under their sum and difference. But the sum is the given line.

COR. 4. The sum of the squares of the whole line and less segment is equal to three squares of the greater segment. For, the square of the sum is equal (*Theor.* 3, *Chap. V.*) to the sum of the squares of the segments with two rectangles under them. Therefore, the sum of the squares of the whole line and less segment is equal to one square of the greater segment, two squares of the less and two rectangles under the segments. But these two squares and two rectangles form double the rectangle under AB and BO_2, which is equal to the square AA_1. Therefore, the sum of the squares of AB and BO is equal to three squares of AO, *as stated.*

COMPOUND RATIO.

COMPOUND RATIO is used in geometry to express the relations of magnitudes as to quantity which cannot be directly represented by a simple ratio. To a certain extent it is an arithmetical measure of this relation, and involves the multiplication of fractions.

Its principle is, that between the two terms of a simple ratio one or more other magnitudes may be introduced, repeated as consequent and antecedent, and thus the ratio be converted into one compounded, for the several simple ratios of which other equal

ratios may be substituted in different terms. Thus, when only one magnitude is introduced, there may be four terms in the compound ratio; when two, there may be six, and so on.

For instance, if the ratio of two magnitudes, A and B, has to be compounded, the magnitude X may be placed between them as consequent to A first, and then antecedent to B; and then we say that A is to B in a ratio compounded of A to X and X to B. Then, again, for these two ratios we may substitute their equals, and say that A is to B in a ratio compounded of P to Q and R to S, these latter ratios being equal, respectively, to those of A to X and X to B.

In like manner, a simple ratio may be converted into one compounded of any number of simple ratios. Between A and B let three magnitudes be placed, repeated as consequents and antecedents. Then, if X, Y, Z be these magnitudes, A is to B, in a ratio compounded of A to X, X to Y, Y to Z, and Z to B; and for these ratios we may substitute others equal to them, but in different terms, such as P to Q, R to S, T to U, and V to W. The rule thus exemplified applies to any number of interposed magnitudes. The result is the same, namely, that in the compounded ratio the traces of the original magnitudes compared sometimes disappears.

DUPLICATE RATIO is a particular case of compound ratio, in which two equal simple ratios are compounded into one; as, for instance, the ratio of a to b compounded with the ratio of a to b. But it is not necessary that the second ratio should be in the same terms, a and b. An equal ratio in different terms may be substituted for it. The two ratios thus compounded being equal, there is a doubling of the *ratio*, not by addition, but by repetition in the compounding. Hence the expression "Duplicate Ratio."

But duplicate ratio may be considered in another point of view. When three magnitudes are in con-

tinued proportion, since the first is to the second as the second to the third, there are two equal ratios, which, if compounded, must give a duplicate ratio, namely, the ratio compounded of the first to the second of the three proportionals and the second to the third. As the second magnitude in the compounding is repeated as consequent and antecedent, it may be dropped out, the result being the simple ratio of the first to the third. But that ratio is the result of the duplication of that of the first to the second. Hence Euclid's definition of duplicate ratio:—

"When three magnitudes are proportional, the first is said to have to the third a duplicate ratio of that which it has to the second."

SUBDUPLICATE RATIO is the ratio obtained by reversing the process described above, going back from a ratio viewed as duplicate to the original ratio from which it could have been produced. Hence the term "*sub*duplicate." And, as such, it may be defined in terms corresponding to that of duplicate ratio, thus:—

When three magnitudes are proportional, the first is said to have to the second a subduplicate ratio of that which it has to the third.

Or, it may be otherwise defined:—

The subduplicate ratio of two magnitudes is the ratio of the first magnitude to their mean proportional.

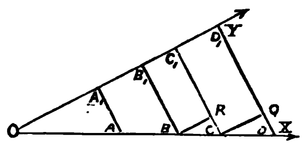
TRIPLICATE RATIO is similar to duplicate, but is compounded of three equal ratios. And, as when four magnitudes are proportional, there are three consecutive equal ratios, the first must be to the fourth in a ratio the triplicate of that of the first to the second. And so on, if there be five proportionals, the first is to

the fifth in the Quadruplicate ratio of the first to the second; and if there be six proportionals there will be a Quintuplicate ratio, and so on.

RATIO EX EQUALI.—This is another compound ratio, which, as regards *lines* arranged in *regular* order in two series, will be best understood by putting its statement into the form of a theorem.

24. *If there be Two Series of Magnitudes, an Equal Number in Each, and the First Magnitude be to the Second in the First Series as the First to the Second in the Second Series, and the second be to the third in the first series as the second to the third in the second series, and so on, each magnitude in the first series to that immediately following as the corresponding magnitude in the second series, &c.; then,* ex equali, *the First is to the Last in the First Series as the First to the Last in the Second Series.*

From the point O of intersection of any two directives X and Y, take on X distances OA, AB, BC, and CD, to represent the magnitudes in order of the first series, and take also on Y other distances OA_1, A_1B_1, B_1C_1, C_1D_1, representing in order the magnitudes of the second series, and let the joining lines AA_1, BB_1, CC_1, and DD_1 be supposed drawn.



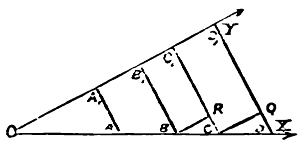
Then, since OA is to AB as OA_1 to A_1B_1, the lines AA_1 and BB_1 (*Theor.* 2) are parallel. Also, since AB is to BC as A_1B_1 is to B_1C_1, the lines BB_1 and CC_1 are parallel; and, since BC is to CD as B_1C_1 to C_1D_1, the lines CC_1 and DD_1 are parallel. Therefore, the lines BB_1, CC_1, and DD_1 are parallel to each other and each to AA_1. But, CQ being a parallel to Y meeting DD_1 in Q, the two triangles OAA_1 and CDQ are equiangular, and therefore OA is to CD as OA_1 is to CQ. But the figure CQD_1C_1 is a parallelogram, and therefore CQ is equal to C_1D_1. Therefore, OA is to CD as OA_1 is to C_1D_1, that is the first magnitude OA is to the last CD in the first series as the first OA_1 is to the last C_1D_1 in the second series, *as stated.*

N.B.—It should be noted, as regards this demonstration, that the equality of the ratios of the first to the last in the two

series depends on the lines AA_1, BB_1, CC_1 and DD_1 being parallel; and that parallelism depends on each pair of the single ratios being equal. If any one pair of the single ratios were not equal, such as AB to BC and A_1B_1 to B_1C_1, the transverse lines BB_1 and CC_1 would not be both parallel to AA_1; and consequently the triangles OAA_1 and CDQ would not be similar, and the ratios of the first and last in the two series not equal.

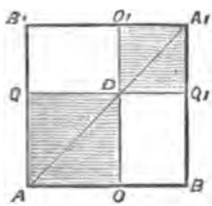
25. *The Duplicate Ratio of two Equal Ratios are equal.*

Using the same figure as before, let the two equal simple ratios be those of OA to AB and OA_1 to A_1B_1, and suppose BC and B_1C_1 to be the third proportionals to OA and AB, OA_1, and A_1B_1 respectively. Then, the duplicate ratios of OA to AB and OA_1 to A_1B_1 are the simple ratios of OA to BC and OA_1 to B_1C_1. But, since OA is to AB as OA_1 to A_1B_1, the transverse lines AA_1 and BB_1 are (*Theor.* 2) parallel. Also, since AB is to BC as A_1B_1 is to B_1C_1, the lines BB_1 and CC_1 are parallel; and, therefore, CC_1 is parallel to AA_1. But, BR being a line parallel to Y meeting CC_1, the figure BRC_1B_1 is a parallelogram, and BR is equal to B_1C_1, and the triangles OAA_1 and BCR are similar. Therefore, OA is to BC as OA_1 is to BR. But BR is equal to B_1C_1. Therefore, OA is to BC as OA_1 is to B_1C_1; and the duplicate ratios of the equal ratios of OA to AB and OA_1 to A_1B_1 are equal, *as stated*.



26. *The Duplicate Ratio of Two Lines is the ratio of their Squares.*

Let AO and BO be the lines placed *in directum*, and ABA_1B_1 the square on AB. Let, also, OO_1 be a parallel through O to AB_1, meeting the diagonal AA_1 in D, and QQ_1 a parallel through D to AB. Thus, the square is divided (*Theor.* 20, *Chap. III.*) into the two squares AD and A_1D of AO and BO about the diagonals and the two rectangles OQ_1 and O_1Q under AO and BO. But the duplicate ratio of AO to BO is (*Def.*) the ratio compounded of AO to BO and AO to BO. But, also, since BQ_1 is equal to AO, and A_1Q_1 equal to BO,

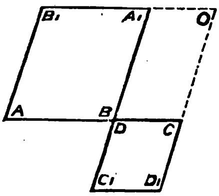


the duplicate ratio of AO to BO is the ratio compounded of AO to BO and BQ_1 to A_1Q_1. But, since QQ_1 is parallel to AB, AO is to BO as the square AD to the rectangle OQ_1; and, since OO_1 is parallel to A_1B, BQ_1 is to Q_1A_1 as the rectangle OQ_1 is to the square A_1D. The rectangle OQ_1, thus occurring as consequent and antecedent, may be dropped out, and the duplicate ratio of AO to BO becomes the ratio of the squares of AO and BO, *as stated.*

REVERSE.—*The Subduplicate Ratio of Two Squares is the ratio of their sides.* For, the subduplicate ratio of the two squares is (*Def.*) the ratio of the first square to the mean proportional to the squares. But this mean proportional (*Theor.* 20, *Cor.* 6) is the rectangle under their sides. Therefore, the subduplicate ratio of the squares AD and A_1D is the ratio of the square AD to the rectangle OQ_1. But this ratio, since the square and rectangle are between the same parallels, AB and QQ_1, is the ratio of AO to BO, the sides of the squares, *as stated.*

27. *The Areas of Equiangular Parallelograms are to each other in a Ratio Compounded of the ratios of their Sides.*

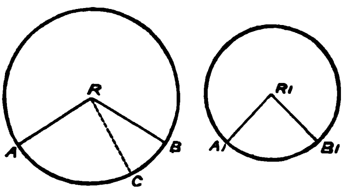
Let ABA_1B_1 and CDC_1D_1 be the equiangular parallelograms, and suppose them so placed that the sides AB and CD be *in directum.* Then, since the angles B and D are equal, the sides A_1B and C_1D are also *in directum.* Let, also, B_1A_1 and D_1C be produced to meet in O. Then the figure $DCOA_1$ is a parallelogram equiangular with AA_1 and CC_1. Place, now, this parallelogram DO for a compound ratio between AA_1 and CC_1; and the ratio of AA_1 to CC_1 becomes a ratio compounded of AA_1 to DO, and DO to CC_1. But AA_1 is to DO (*Theor.* 13) as AB to CD; and DO is to CC_1 as A_1B to C_1D. Therefore, the parallelograms AA_1 and CC_1 are to each other in a ratio compounded of AB to CD and A_1B to C_1D, *as stated.*



COR. 1. Hence, if the angles B and D become right angles, the ratio compounded of AB to CD and A_1B to C_1D becomes the ratio of the rectangle under AB and A_1B to the rectangle under CD and C_1D; and, consequently, as regards lines, *the ratio compounded of two given ratios is the ratio of the rectangle under the antecedents to the rectangle under the consequents.*

COR. 2. Hence, also, since triangles may be considered the halves of parallelograms, *if two triangles have an angle in each equal their areas are proportional to the rectangles under their sides about the equal angles.*

28. *Arcs of Unequal Circles are to each other in a Ratio Compounded of the ratios of the Angles subtended by them at their centres and of their Circumferences.*



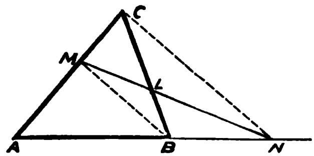
Let AB, A_1B_1 be the two arcs of the circles R and R_1, and RC a radius of the larger circle, making, with RA, an angle ARC equal to the angle $A_1R_1B_1$ in the circle R_1. Then, the arc AB is to the arc A_1B_1 in a ratio compounded of AB to AC, and AC to A_1B_1. But the ratio of arc AB to arc AC is equal to the ratio of the angle ARB to the angle ARC; and the ratio of the arcs AC and A_1B_1, their central angles being equal, is (*Theor.* 15) the ratio of the circumferences of R and R_1. Therefore, the ratio of the arc AB to the arc A_1B_1 is equal to the ratio compounded of the ratios of the angles ARB and $A_1R_1B_1$, and the circumferences of the circles R, R_1, *as stated.**

29. *If a Transversal cut the Three Sides of a Triangle, Two Internally and One Externally, the Segments of that cut externally are to each other in a ratio compounded of the ratios of the segments of the other two sides.*

Let ABC be the triangle cut by the transversal MLN, and BM and CN joining lines. Then, since the triangles AMN and BMN have a common vertex M, and their bases are *in directum*, these bases are to each other as their areas. That is, AN is to BN as the triangle AMN is to BMN. Put CMN between these two triangles for a compound ratio, and AN will

* For the reason given in the foot-note, page 125, this compound ratio may be stated as a ratio compounded of the ratios of the central angles and of the radii of the circles.

be to BN in the ratio compounded of AMN to CMN, and CMN to BMN. But AMN is to CMN as their bases AM and CM; and CMN is to BMN (*Theor.* 16) as CL is to BL; that is, as the segments into which the line CB joining their vertices is divided at L. Therefore AN is to BN in the ratio compounded of AM to CM, and CL to BL, *as stated.*



The demonstration is exactly the same when the three sides are all cut externally.

30. *If Three Directives through the Vertices of a Triangle meet in a Point and cut the Opposite Sides, the segments of any one side are to each other in a ratio compounded of the ratios of the segments of the other two sides.*

Let ABC be the triangle (Fig. 1), and AL, BM, and CN the three directives meeting at O. Then, since AB is the line joining the vertices of the two triangles AOC, BOC, standing on the common base OC, the ratio of AN to BN is (*Theor.* 16) the ratio of the triangles AOC and BOC. Place the triangle AOB between these triangles for a compound ratio, and AN will be to BN in a ratio compounded of AOC to AOB, and AOB to BOC. But AOC is to AOB (*Theor.* 16) as CL to BL, and AOB to BOC as AM to CM. Therefore AN is to BN in the ratio compounded of AM to CM and CL to BL, *as stated.*

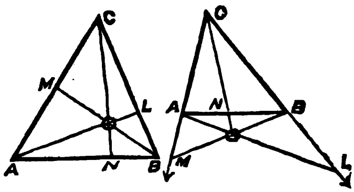
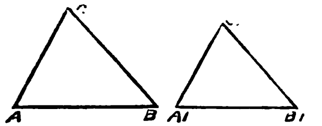


Fig. 1. Fig. 2.

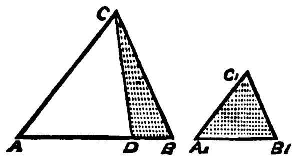
The second Figure illustrates the above demonstration for the case of the point O being outside the triangle when two of the sides CA and BC are cut externally.

31. *The Areas of Similar Triangles are to each other in the Duplicate Ratio of their homologous Sides.*

Let ABC and $A_1B_1C_1$ be the two similar triangles. Then, being the halves of equiangular parallelograms, they are to each other in a ratio compounded (*Theor.* 27) of the ratios of their sides about the equal angles C and C_1, that is, in a ratio compounded of AC to A_1C_1, and BC to B_1C_1. But, the triangles being similar, the ratio of BC to B_1C_1 is equal (*Theor.* 3) to that of AC to A_1C_1. The compound ratio thus becomes the ratio compounded of the equal ratios of AC to A_1C_1, and AC to A_1C_1, that is, the duplicate ratio of the homologous sides AC and A_1C_1, *as stated.*



EUCLID'S PROOF.—Let ABC and $A_1B_1C_1$ be the similar triangles, and CD a line from the vertex of ABC cutting off from AB a segment BD, a third proportional to AB and A_1B_1. Then, AB is to A_1B_1 as A_1B_1 is to BD. But AB is also to A_1B_1 as BC is to B_1C_1. Therefore BC is to B_1C_1 as A_1B_1 is to BD. The two triangles $A_1B_1C_1$ and BCD consequently, having the angles B and B_1 equal, and the sides about these equal angles reciprocally proportional, are (*Theor.* 17, *Cor.* 1) equal in area.

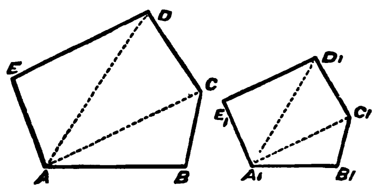


The ratio, therefore, of the triangle ABC to $A_1B_1C_1$ is equal to that of ABC to BCD. But these triangles, having a common vertex C, and standing on portions of the same directive AB, are (*Theor.* 13) to each other in the ratio of AB to BD, that is, in the ratio of AB to the third proportional to it and A_1B_1, or in the duplicate ratio of the homologous sides, AB and A_1B_1, *as stated.*

32. *Similar Polygons may be Divided into Similar Triangles, equal in number; and their Homologous Diagonals are proportional to their homologous sides.*

Let $ABCDE$ and $A_1B_1C_1D_1E_1$ be the similar polygons, and from any pair of corresponding points, A and A_1, suppose the diagonals AC, AD and A_1C_1, A_1D_1 drawn, dividing the polygons into triangles. Then, since in the triangles ACB and $A_1C_1B_1$ the angles B and B_1 are equal, and the sides AB and BC, A_1B_1 and B_1C_1 are proportional, the triangles (*Theor.* 4) are

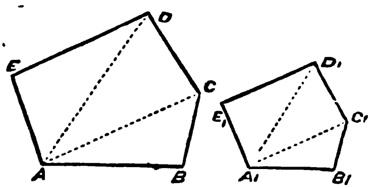
similar; and the diagonals AC and A_1C_1 are to each other in the ratio of the sides AB and A_1B_1; and, also, the angles ACB and $A_1C_1B_1$ are equal. Take these equal angles from the angles BCD and $B_1C_1D_1$, and the remainders, the angles ACD and $A_1C_1D_1$, are equal. And therefore,



in the two triangles ACD and $A_1C_1D_1$, the angles at C and C_1 are equal, and the sides, AC and A_1C_1, are proportional to CD and C_1D_1. Therefore the triangles are similar, and the diagonals AD and A_1D_1 are proportional to DC and D_1C_1, and therefore to AB and A_1B_1. And so on, all round the polygons, it can be proved that they can be divided into similar triangles, and that the diagonals are proportional to any pair of homologous sides, *as stated.*

33. *The Areas of Similar Polygons are to each other in the Duplicate Ratio of their homologous sides.*

Let $ABCDE$ and $A_1B_1C_1D_1E_1$ be the two similar polygons divided into similar triangles, as in the last theorem. Then, since each pair of homologous triangles, one from each polygon, are to each other in the duplicate ratio of their homologous sides, they are in the dupli-



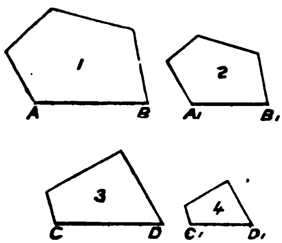
cate ratio of any one pair of such sides, such as AB and A_1B_1. The sum, therefore, of all the antecedent triangles, the first polygon, is to the sum of all the consequent triangles, the second polygon, in the duplicate ratio of AB to A_1B_1, or of BC to B_1C_1, or of any other pair of homologous sides of the two polygons, *as stated.*

COR. 1. Hence, since two squares are similar figures, each divisible into two similar triangles, it follows, as has already been (*Theor.* 26) proved, that they are to each other in

the duplicate ratio of their sides, or that the duplicate ratio of two lines is the ratio of their squares.

34. *If Four Lines be Proportional, the Similar rectilineal Figures described on them are proportional.*

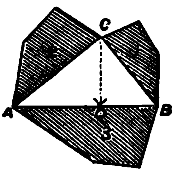
Let AB, A_1B_1, CD, C_1D_1 be the four proportionals. Then the similar figures, Nos. 1 and 2, on AB and A_1B_1 are (*last Theor.*) to each other in the duplicate ratio of AB to A_1B_1. So, likewise, the similar figures, Nos. 3 and 4, on CD and C_1D_1, are to each other in the duplicate ratio of CD to C_1D_1. But the duplicate ratios of two equal ratios (*Theor.* 25) are equal. Therefore the figures on AB and A_1B_1 are proportional to the figures on CD and C_1D_1, *as stated.*



N.B.—It should be noted that the figures 1 and 2, though necessarily similar to each other, are not required to be similar to figures 3 and 4, which, however, must be similar to each other, though of a different outline of angles and sides from figures 1 and 2.

35. *The Sum of two Similar rectilineal Figures, Similarly Described on the sides of a Right-angled Triangle, is equal to the similar rectilineal figure similarly described on the Hypothenuse.*

Let ACB be the right-angled triangle, and AB the hypotenuse; and let the shaded figures, Nos. 1, 2, and 3, be the similar rectilineal figures on the sides and hypothenuse. Then the figures Nos. 3 and 2, being similar, are to each other (*Theor.* 33) in the duplicate ratio of AB to BC; which, since BD is (*Theor.* 11) the third proportional to AB and BC, is equal to the ratio of AB to BD. In like manner, it is proved, since AD is the third proportional to AB and AC, that the figure No. 3 is to figure No. 2 as AB to AD. Therefore the figure



No. 3 is to the sum of the figures Nos. 1 and 2 as AB is to the sum of AD and BD; that is, as AB is to AB—a ratio of equality, *as stated.*

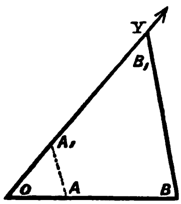
This may otherwise be thus proved. The figure No. 3 is to figure No. 1 in the duplicate ratio of AB to BC, that is (*Theor.* 26), in the ratio of the squares on AB and BC. So, likewise, the figure No. 3 is to figure No. 2 as the square on AB is to the square on AC. The figure No. 3 is, therefore, to the sum of figures Nos. 1 and 2 as the square on AB is to the sum of the squares on AC and BC. But the sum of the squares Nos. 1 and 2 is equal (*Theor.* 14, *Chap. V.*) to the square No. 3. Therefore the ratio of No. 3 to the sum of Nos. 1 and 2 is a ratio of equality, *as stated.*

If the similar figures become similar polygons inscribed in semicircles described on the three sides of the triangle, and the sides in each, excepting the homologous sides which coincide with the sides of the triangle, increase indefinitely in number, decrease in magnitude, and are all equal to each other, the polygons tend to become equal in area to the semicircles; and, when the sides become infinitely small, eventually become such. Hence it is inferred that the semicircle on the hypothenuse of a right-angled triangle is equal to the sum of the semicircles on the sides.

PROBLEMS.

We now enter, by the introduction of ratio and proportion, on the widest possible field for the solution of problems on the elementary geometry of the directive and circle. The fundamental problems are but a few, mostly taken from Euclid; but a large number of exercises will be given at the close of the chapter embracing all the principles which it has been the purpose of this treatise to demonstrate. The elementary problems are those which follow:—

1. *To divide a given line OB in a given ratio.*



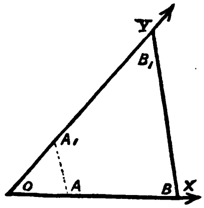
From the extremity O of OB draw at any angle a directive Y; and take on it, commencing at O, two distances OA_1 and A_1B_1 equal to the given lines representing the given ratio. Then, join B_1 with B, and from A_1 draw A_1A parallel to B_1B; and OB will be cut at A in the required ratio.

For, since AA_1 and BB_1 are parallel, the segments OA and AB of OB are (*Theor.* 1) to each other in the same ratio as the segments OA_1 and A_1B_1 of Y are, that is, in the given ratio; and OB is divided, *as required.*

Cor. 1. Hence, from any given line OB any assigned part may be cut off, such as a seventh part or an eleventh. The line OB has to be cut in a given ratio. And all that is necessary is to take on Y, any distance OA_1, and also another distance OB_1, the same multiple of OA_1, as OB is required to be of the part to be cut from it. Then, on joining B_1 with B, and drawing a parallel A_1A to BB_1, the point A in which A_1A cuts OB is the point of section required, and OA the *required part cut off.*

2. *To find a fourth proportional to three given lines.*

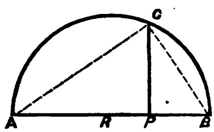
Let X and Y be two directives meeting at O; and take on the directive X two distances, OA and AB, equal respectively to the first and second lines; and then take on Y the third given line OA_1. Join, then, A_1 with A, and from B draw (*Prob.* 10, *Chap. II.*) a parallel BB_1 to AA_1; and A_1B_1 will be the required fourth proportional.



For, the lines AA_1 and BB_1 being parallel to each other, OA is (*Theor.* 1) to AB in the ratio of OA_1 to A_1B_1; and therefore A_1B_1 is the fourth proportional, *as required.*

3. *To find a mean proportional to two given lines.*

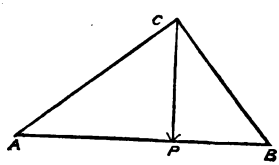
Let AP and BP be the two given lines placed *in directum* forming the line AB. Bisect AB at R, and, with R as centre and RA as radius, describe a semicircle ACB on AB. Erect, then, at P a perpendicular CP to AB meeting the semicircle at C. Then CP is the required mean proportional.



For, since the triangle ACB stands on a diameter AB of the circle R, the angle at C is (*Theor.* 12, *Chap. II.*) a right angle; and therefore the triangle ACB is a right-angled triangle. But (*Theor.* 11) CP is a mean proportional to AP and BP. It is, therefore, the mean proportional *required.*

4. *To find a third proportional to two given lines AP and CP.*

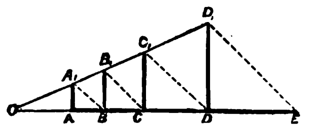
Place the two lines AP and CP at right angles to each other, and join A with C. Erect, then, a perpendicular at C to AC to meet AP externally at B. Then BP is the third proportional required.



For, since the triangle ACB is right-angled at C and CP is perpendicular to AB, the two triangles APC and BPC are similar (*Theor.* 11) to each other, and CP is a mean proportional to AB and BP. Therefore, BP is the third proportional to AP and CP, the two given lines, *as required.*

5. *Given the first and second terms of a series of magnitudes increasing in continued proportion, to find any assigned number of terms and their sum.*

Let AB be the given first term of the series, and erect at A and B lines AA_1 and BB_1 perpendicular to AB, the former equal to AB and the latter equal to the given second term of the series. Join, then, A_1 and B_1, and produce the joining line A_1B_1 both ways, on one side indefinitely, and on the other to meet BA externally at O. Join, then, A_1 with B, and from B_1 draw a parallel B_1C to A_1B to meet the directive OE in C. At C then erect a perpendicular CC_1 to OE to meet OD_1 in C_1, and from C_1 draw a parallel C_1D to A_1B to meet OE in D, and then erect again a perpendicular DD_1 to OE meeting OD_1 in D_1. And, so on, continue the same construction until the perpendiculars CC_1, DD_1, &c., are in number equal to the required terms—drawing from the extremity D_1 of the last perpendicular (suppose DD_1) a parallel D_1E to A_1B to meet OE in E. Then, the perpendiculars AA_1, BB_1, CC_1 and DD_1, &c., will be all the terms of the series, and AE their sum.

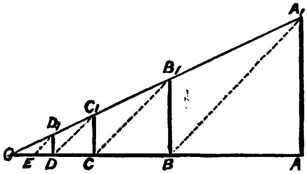


For, since AA_1 and BB_1 are parallel, AA_1 (*Theor.* 1) is to BB_1 as OA to OB. But OA is to OB as OA_1 is to OB_1, or, since A_1B and B_1C are parallel, as OB is to OC. Therefore OA is to OB as OB to OC, and OA, OB, and OC are three proportionals. But the ratio of OB to OC, since BB_1 and CC_1 are parallel, is equal to that of OB_1 to OC_1; which again, since B_1C and C_1D are parallel, is the ratio of OC to OD. Hence, OA, OB, OC, and OD are four continued proportionals; and, since the triangles with their vertices at O are equiangular, AA_1, BB_1, CC_1, and DD_1 are in continued proportion, and CC_1 and DD_1 are two of the terms of the series of which AA_1 and BB_1 are the first and second terms. And it is evident that, by continuing the construction, other terms may in like manner be had, each greater than the one preceding.

Also, since AB is equal to AA_1, the triangle ABA_1 is a right-angled isosceles triangle, and the line A_1B makes half a right angle with OE. Therefore, the lines B_1C, C_1D, and D_1E make each half a right angle with OE; and consequently BC is equal to BB_1, CD to CC_1, and DE to DD_1, and AE is the sum of the four proportionals, *as required.* And the same construction holds good for finding the sum of any greater number of terms. But, since the series is one of increasing proportion, the sum of only a *finite* number of terms can be obtained.

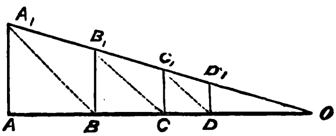
6. *Given the first and second terms of a series of magnitudes decreasing in continued proportion, to find any assigned number of terms, and their sum.*

The construction and proof, in this problem, are the same as in the last. AB is the first term, and AA_1 is equal to AB, and BB_1 is the second term. The dotted lines A_1B, B_1C, C_1D, &c., are all parallel to each other, and make each half a right angle with AO. Also, CC_1 and DD_1 (in the Figure) are two of the proportionals so far found; and AE is the sum of the four terms. If more than the four terms be required, the same construction must be repeated until the assigned number of terms be obtained. In this case, however, there is no limitation of the number of terms, the sum of which may be obtained; as will be evident from the next problem.



7. *Given the first and second terms of a series of magnitudes decreasing* ad infinitum *in continued proportion, to find their sum.*

Let AB be the first term, and erect at A and B, as in the preceding problems, perpendiculars AA_1 and BB_1 equal to AB and the given second term. Join, then, A_1 with B_1, and produce A_1B_1 to meet AB externally at O. Then AO is the required sum of the series. For:—



1st. Since, after the summation AD of the series up to D, a distance has to be taken from D on DO equal to DD_1, and added to AD, and this addition has to be repeated with all the succeeding terms, the extremity of the sum up to any term of the series must progressively approach closer to O and eventually coincide with it.

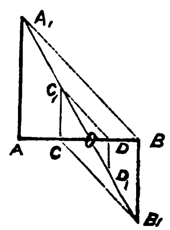
2nd. But the sum cannot extend beyond O. For, since AA_1 is equal to AB, the ratio of AA_1 to AO is a ratio of "lesser inequality," that is, a ratio in which the antecedent is less than the consequent. Therefore, since the triangles AOA_1, BOB_1, COC_1, &c.,

are all similar, any term such as DD_1 must be less than DO; and, when added to the sum AD of the preceding terms, it must leave a remainder on DO. And this must hold good for every term added on in the successive summation to the end. The sum, therefore, cannot exceed AO, but must be equal to AO, *as stated.*

COR. 1. Hence it may be shown that the sum of the series is a third proportional to the difference of the first and second terms and the first term. For, since the triangles AOA_1 and BOB_1 are similar, AO is to BO (*Theor.* 3) as AA_1 is to BB_1; and, therefore (*by division*), AO is to the difference of AO and BO as AA_1 is to the difference of AA_1 and BB_1. But the difference of AO and BO is the first term AB, and AA_1 is also the first term. Therefore the sum AO is to AB as AB is to the difference of AA_1 and BB_1, that is, to the difference of the first and second terms, *as stated.*

8. *Given the first and second terms of a series of magnitudes decreasing in continued proportion, to find their sum, every second term being subtracted from the preceding term.*

Let AB be the first term. Erect, then, at A a perpendicular AA_1 to AB equal to AB. At B, then, draw downwards BB_1 equal to the second term, and also perpendicular to AB. Join, then, A_1 with B_1; and the distance AO cut off from AB is the sum required.



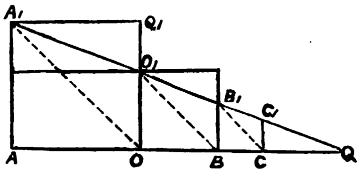
For, since AA_1 is equal to AB, the dotted line A_1B makes half a right angle with AB; and the lines B_1C_1 and C_1D, parallel to it, also make each half a right angle with AB. Therefore, BC is equal to the second term B_1B, and AC is the difference of the first and second terms. But, since AA_1 is greater than AO, the line BC, which is equal to BB_1, is greater than BO, and C must lie to the left of O. In like manner, on taking CD equal to CC_1, the point D must lie to the right of O; and then, again, on taking to the left from D a distance equal to DD_1, its extremity must lie to the left of O; and the difference between the third and fourth terms will be added to AC, and give the sum of the first four terms. And so on, the process of adding on to the sum obtained at any stage the difference of the odd and even terms, may be continued, *ad infinitum*, until the extremity of the sum reaches O without ending on either side of it. Hence AO is the sum of the series, equal to the difference of the sums of the odd and even terms, *as required.*

Thus, if the second term be half of the first, the sum is two-thirds of the first.

COR. 1. Hence it is evident, as in the last problem, that the sum of the series is a third proportional to the sum of the first and second terms and the first term. For, since the triangles AOA_1 and BOB_1 are similar, AO is to BO as AA_1 is to BB_1. Therefore (*by composition*) AO is to the sum of AO and BO as AA_1 is to the sum of AA_1 and BB_1. But the sum of AO and BO is the first term AB, and AA_1 is also the first term. Therefore, the sum AO is to AB as AB is to the sum of AA_1 and BB_1, that is, to the sum of the first and second terms, *as stated.*

9. *To find the sum,* ad infinitum, *of the greater segments of a given line AB cut consecutively in extreme and mean ratio.*

Let O be the point at which the given line AB is cut in extreme and mean ratio, and OA_1 and BO_1 the squares on the segments AO and BO. Join, then, A_1 with O_1, and produce A_1O_1 to meet AB externally at Q. Then AQ is the sum of the greater segments, *ad infinitum*, and is equal to the sum of the whole given line and its greater segment.



For, since AO and AA_1 are each equal to the greater segment of AB, and OO_1 is the greater segment (*Theor.* 23) of AO cut in extreme and mean ratio, the distance AQ (*Prob.* 7) is the sum of the greater segments. But, BB_1 is the greater segment of OB, cut in extreme and mean ratio, and is equal to O_1Q_1. Therefore, since the triangles BB_1Q and $Q_1O_1A_1$ are equiangular, and have their sides BB_1 and O_1Q_1 equal, their other sides (*Theor.* 10, *Chap. III.*) are equal; and consequently BQ is equal to A_1Q_1, that is, equal to the greater segment AO. Hence, the sum of the series is equal to the sum of the whole line AB and the greater segment AO, *as stated.*

COR. 1. Hence it is evident that the sum of the less segments, *ad infinitum*, is the line OQ, which is evidently equal to the whole line AB.

10. *To construct on a given line AB a polygon similar to a given polygon* $A_1B_1C_1D_1E_1$.

Divide the given polygon into the triangles $A_1B_1C_1$, $A_1C_1D_1$, $A_1D_1E_1$, &c., and describe on AB a triangle ABC equiangular with $A_1B_1C_1$. Then, on the side AC of the triangle ACB so constructed, describe a triangle ACD equiangular with $A_1C_1D_1$; and so on, describe on AD a triangle ADE equiangular with $A_1D_1E_1$, repeating the construction for the requisite number of sides. Then, the figure $ABCDE$, so constructed, is one of the required polygons, being divisible into as many similar triangles as $A_1B_1C_1D_1E_1$ is divided into, and having (*Theor.* 32) its homologous sides and diagonals proportional to those of that polygon.

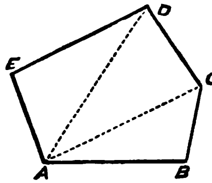


Fig 1.

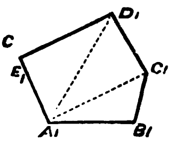
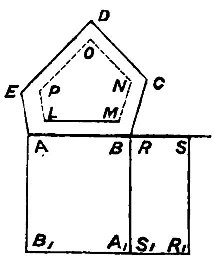


Fig. 2.

N.B.—It should be noted that, since, in the construction, AB may be made homologous to any side of the given polygon as well as A_1B_1, there are as many polygons that may be constructed on AB, all similar to each other, but not similarly placed, as the given polygon has sides, all of different magnitudes, unless the given polygon has equal sides. And the constructed polygon for which the least side of the given one is made homologous to AB, will be the largest polygon, and that for which the greatest side is made homologous to AB will be the smallest polygon.

11. *To construct a polygon of a given area similar to a given polygon ABCDE.*

On any side AB of the given polygon construct (*Prob.* 2, *Chap. IV.*) a rectangle ABA_1B_1 equal to it in area. Then, on the side A_1B of this rectangle construct another RSR_1S_1 equal to the area given of the required polygon. Find, then, a mean proportional LM to AB and RS, and describe on it (*last Prob.*) a polygon, $LMNOP$, similar to the given one, and having LM homologous to AB. The polygon so described on LM is the one required.



For, since the polygon on LM is to that on AB in the duplicate ratio of LM to AB, it is to that on AB in the simple ratio of RS to AB,

or in the ratio of the rectangle RR_1 to AA_1. But the rectangle AA_1 is, by construction, in area equal to $ABCDE$. Therefore, the areas of the rectangle RR_1, and the polygon $LMNOP$, have the same ratio to the equal figures ABA_1B_1 and $ABCDE$; and, consequently (*Ax.* 5, *Chap. VI.*), are equal to each other; and the polygon on LM is *the one required.*

EXERCISES.

1. Draw a directive through a given point across two diverging directives so that its segments, measured from the point, cut off by the two directives have a given ratio.
2. Draw a directive from a given point across three directives diverging from another given point, so that its intercepted segments have a given ratio.
3. Draw from a given point a directive across two diverging directives so that their segments cut off from their point of divergence have a given ratio.
4. Do the same, so that the rectangle under the segments cut off from their points of divergence be a given magnitude.
5. Do the same, so that the area of the triangle formed by the intersection of the three directives be a given magnitude.
6. From two given points draw two lines meeting on a given directive, and having a given ratio.
7. Draw the same from two given points to meet on a circle.
8. Given the ratio of two lines and the rectangle under them, find the lines.
9. Given the ratio of two lines and the sum of their squares, find the lines.
10. Given the ratio of two lines and the difference of their squares, find the lines.
11. Given the ratio of two lines and their sum or difference, find them.
12. Given the base, area, and ratio of the sides of a triangle, construct it.
13. Given base, vertical angle, and ratio of the sides of a triangle, construct it.
14. Given base, sum of squares of sides and ratio of sides of a triangle, construct it.
15. Through a point within a circle to draw a chord so that its

segments at the point have a given ratio. Do the same when the point is without the circle.

16. Prove that the square of the sum of any line and the less segment of that line cut in extreme and mean ratio is equal to five squares of the greater segment of the line.

17. If a line two feet in length be cut in extreme and mean ratio, find the length of its greater segment.

18. If the first term of a series of magnitudes decreasing *ad infinitum* in continued proportion be double the second term, what will be the magnitude of the sum of the series?

19. Find the sum of the series if the second term be five-sevenths of the first.

20. If a line drawn from a given point to a given directive be cut in a given ratio, the *locus* of its point of section is a directive parallel to the given one.

21. If a line drawn from a given point to a given circle be cut in a given ratio, the *locus* of the point of section is a circle. Find its centre and radius.

22. Prove that the directive through the given point touching one of these circles also touches the other circle.

23. Draw a common tangent to two given circles, and prove that, if the two circles do not intersect, there are four common tangents; and that these tangents in pairs meet at the points at which the line joining the centres of the circles is cut internally and externally in the ratio of the radii of the circles.

24. Prove, also, that all lines drawn through either of these points (which are termed "the centres of similitude" of the two circles) and terminated, each on the portions of the circles both convex to that point, or both concave to it, are cut at the point in the ratio of their radii.

25. Inscribe a square in a triangle, one of its sides being on the base of the triangle; and prove that the rectangle under the side of the inscribed square and the sum of the base and perpendicular on the base from the vertical angle is double the area of the triangle.

26. If two lines be drawn from the vertex of a triangle inscribed in a circle making equal angles with its sides, one terminated by the base and the other by the circle, the rectangle under these lines is equal to the rectangle under the sides of the triangle.

27. A triangle being inscribed in a circle, prove that the line drawn from the point of bisection of the lower arc of which the base is a chord to either extremity of the base is equal to the distance of the centre of the circle *inscribed* in the triangle from that point, and also equal to the distance of the centre of the circle *exscribed* to the base.

28. Prove, also, that the distance of the point of bisection

of the upper arc of the circle from either extremity of the base is equal to the distances from that point of the centres of the two circles *exscribed* to the two sides.

29. Hence, prove that, being given base and vertical angle of a triangle, the locus of the centres of the inscribed circle, and the circle exscribed to the base, is a circle, the centre of which is the point of bisection of the arc of which the base is chord, and radius the distance of that centre from the extremity of the base.

30. Hence, also, prove that, with the same data, the locus of the centres of the two circles exscribed to the two sides of the triangle is a circle, having the bisection of the upper arc of the circumscribing circle for centre, and the distance of that point from either extremity of the base for radius.

31. Prove, also, that the rectangle under the distances of the centres of the inscribed circle and the circle exscribed to the base from the vertex of the triangle, is equal to the rectangle under the sides of the triangle.

32. Prove, in like manner, that the rectangle under the distances of the centres of the two circles exscribed to the sides from the point of bisection of the upper arc of the circumscribing circle is equal to the rectangle under the sides.

33. Prove that the angles subtended at the centres of the two circles exscribed to the sides of a triangle by the base are equal each to half the vertical angle of the triangle.

34. Prove that the angle subtended by the base at the centre of the circle exscribed to the base is equal to half the sum of the base angles of the triangle.

35. The sum of the radii of the two circles exscribed to the sides is equal to double the perpendicular from the bisection of the upper arc of the circumscribing circle on the base.

36. The difference of the radii of the circle exscribed to the base and the inscribed circle is equal to double the perpendicular from the bisection of the lower arc on the base.

37. The sum of the four radii of the three exscribed circles is equal to the sum of four times the radius of the circumscribing circle and one radius of the inscribed.

38. The sum of the squares of the four lines drawn from the centre of the circumscribing circle of a triangle to the centres of the exscribed and inscribed circles is equal to three squares of the diameter of the circumscribing circle.

39. If a circle be circumscribed to a triangle, and from the point of bisection of the arc of which the base is chord a perpendicular be drawn to the greater side of the triangle, it will divide that side into two segments equal to the half sum and half difference of the sides.

40. If the perpendicular be dropped from the point of bisection

of the upper arc on the same side, it will also divide that side into two segments equal to the half sum and half difference of the sides.

41. Construct a triangle, being given its base, vertical angle, and sum of sides.

42. Construct a triangle, being given its base, vertical angle, and difference of sides.

43. Given the base, vertical angle, and radius of inscribed circle, construct the triangle.

44. Prove that the portion of the base of a triangle included between the points of contact with the base of the inscribed circle and the circle exscribed to the base, is the difference of the two sides of the triangle.

45. Given the difference of the sides of a triangle, and the radii of the inscribed circle and of the circle exscribed to the base, construct it.

46. Prove that, if the external bisectors of the three angles of a triangle be produced to meet the opposite sides, the three points of intersection will lie on a directive.

47. If a triangle given in species (that is, having given angles) has one of its vertices at a fixed point, and another vertex moves along a directive, prove that the third vertex will move along another directive.

48. If, under the same circumstances, the second vertex moves along a circle, prove that the third vertex will move along another circle.

49. If m and n be two integer numbers, and a line AB be cut at a point O so that AO be to BO *inversely* as m to n, prove that the sum of m times the perpendicular from A and n times the perpendicular from B on *any* directive is equal to $(m + n)$ times the perpendicular from O on the same directive.

50. If AB be cut *externally* at O, the difference of m and n times the perpendiculars from A and B on the directive is equal $(m - n)$ times the perpendicular from O.

APPENDIX.

Note 1.—Direction.

The introduction into Elementary Geometry of a new idea, proposed essential as a basis of the science, can be justified only on the grounds of its being naturally recognized by the mind as a constituent part of our conception of space, and being, moreover, necessary to carry geometric reasoning to its legitimate conclusions. On both these grounds, the term "direction" has been used in this treatise to denote an idea overlooked by Euclid, namely, that of the quality which constitutes "straightness," in a right line, or the "evenness" with which he describes it to lie between any two of its points. "Straightness" and "evenness" are words in common use, well enough understood in practice; but, for the purposes of an abstract science, they must be limited to a distinct conception. And there is none so real, and so well adapted for that purpose, as *sameness of direction* throughout the whole extension of the line, whether it be finite or indefinite.

As to the first ground mentioned above, by which the introduction of this idea may be justified, it is sufficient to show that direction is essential to our conception of space, and of a right line. Assign to any particular point of space a central position; there is around that point—which may be considered the place of the spectator—an infinitude of directions, which are neither all above nor all below, neither all to the right nor all to the left, but all equally around. In each such direction a trace may be supposed made by the motion of a point; and each such trace is a right line, whether it be finitely extended or indefinitely in opposite directions. In the former case the line has finite length, and may be termed a "finite right line," or simply "line." In the latter, although the idea of length enters, *sameness* of direction throughout is its distinctive quality; and no better term can be found to denote it than "Directive," as a substitute for the old term "Indefinite Right Line."

Further, if there be an infinitude of directions around any one point of space, since all points of space are similar and similarly placed, there must be a corresponding infinitude of directions

around every other point. Also, to every one direction, from the first point, there must be a direction corresponding from every other point; and this affords ground for including all such corresponding directions, and the lines that represent them, under a common denomination. Suppose the term "parallelism," be used to denote this quality; then parallel lines, or parallel directives, should be defined :—directives which have the same direction; the immediate consequence of which is, that parallel directives can never meet; two right lines can intersect in only one point; and only one parallel, or one perpendicular, can be drawn through a point to a given line.

Now, all this leads, without assuming Euclid's fifth postulate, by no means self-evident, to a simple proof of his Twenty-ninth Proposition, such as is given in this treatise. And this brings us to the second ground on which the introduction of this idea into geometry may be justified, namely, that it is *necessary* for the purpose of carrying geometric reasoning to its legitimate results. The question, then, is, did Euclid ever *prove* that proposition? The answer must be :—He did not.

After fixing the foundation of the geometric edifice on his Fourth Proposition, he advanced, without a flaw in the chain of reasoning, and after proving one converse, to his seventeenth proposition. He proved that proposition; and the next step was to prove its converse. A difficulty lay in the way. The converse could not be proved without a true theory of parallel lines derived from his definition that they "never meet." At that stage of his progress the required theory was, on his own principles, not at hand; and the converse was left unproved. He then advanced safely to his twenty-seventh and twenty-eighth propositions, and successfully proved that, if a line make equal angles with two other lines, these lines are parallel, that is, "never meet." But a question presented itself. These lines being proved parallel, will every other line drawn across them cut them at equal angles? This distinctive property had yet to be proved; and he was led to his Twenty-ninth Proposition; which he proved by his fifth postulate, which was the converse of his seventeenth proposition, which again could not be proved without the aid of the twenty-ninth, from which he had started. The reasoning was a palpable "vicious circle," the *petitio principii* of Logic. It would have been much better had he assumed the twenty-ninth itself as his axiom. It is far more self-evident than his fifth postulate, and is itself the very expression of "direction;" and, had he assumed it earlier, he would have proved his seventeenth, and its converse, and met no further difficulty in his "ELEMENTS."

But it has been often said that Euclid's twenty-seventh proposition, combined with an assumed principle that only one

parallel to a line can be drawn through a point, proves his twenty-ninth proposition. So it would, were the principle self-evident. But Euclid himself saw that, at the stage of his progress, up to the end of his twenty-eighth proposition, this principle was by no means self-evident, nor a necessary consequence of his definition of parallel lines. Once proved that a transversal cuts parallel lines at equal angles, it *is* evident, but not before. Since, therefore, Euclid never proved this proposition, and something must be assumed, either the proposition itself must be assumed as self-evident, or the principle of direction admitted from the very commencement of geometric demonstration. There is no other alternative.

In confirmation of the above remarks may be adduced the objection to Euclid's proof of his Twenty-ninth Proposition made, in the fifth century, by Proclus, a great mathematician and an admirer of Euclid. In reference to the Fifth Postulate, to which the proof is reduced, he observes, "But, in our exposition of things prior to theorems, we have asserted that this postulate is not allowed by all to be indemonstrably evident. For, how can this be the case when its converse is delivered among the theorems as demonstrable? For the theorem" (the seventeenth) "which says that the two internal angles of every triangle are less than two right angles, is the converse of this postulate."

That he also had an insight of the root of Euclid's difficulty is evident from the account he gives of the difference between intersecting and parallel lines, namely, that "intersectors diverge infinitely from their point of intersection, and parallels do not." The writer of the article on Geometry in Brewster's "Encyclopædia," clearly seeing the force of Proclus's objection, has made the Twenty-ninth Proposition itself the axiom on which to base the proofs of the propositions given in that article.

There is another point in connection with this question which deserves to be carefully considered. It is commonly stated, and very generally believed, that Geometry, as an abstract science, is based upon *definitions.* This cannot be the case; for the fundamental ideas of space, and point (and direction?), are undefinable, being simple ideas. The best confutation of the prevalent opinion is the following quotation from the "Penny Cyclopædia:"—

"Those who would found geometry upon definitions entirely, may think that the difficulty of the theory of parallels arises from insufficient definition; but those who believe it to be deducible from real and positive convictions, having nothing arbitrary about them, must suspect that in this purely negative definition of parallels, we have not sufficiently described that very obvious relation of position which distinguishes parallelism from convergence, however short be the lines we image to

ourselves, or however little we think of what will take place if they are produced."

Note 2.—Problems.

Problems are separated in this treatise from theorems on the ground that they contribute nothing to the validity of the demonstration of theorems, but impede demonstration, and force the theorems out of a natural order of sequence. The objection is as old as the earliest centuries of the Christian era, and were strongly urged by Arnauld. A theorem is a proposition which states that a certain figure, as defined, whose boundaries are lines, curved or straight, which have only length, no breadth, possesses certain properties. No drawing on board, skin, or paper, can accurately represent this figure. It is an ideal; and the figure on skin or paper is only a rude picture of the ideal which is in the mind—a mere help to the memory to keep attention fixed on the steps of the demonstration. The reasoning, therefore, in the proof of a theorem is purely hypothetic, that is, intended to prove that, if the figure on the paper be truly a triangle, as defined, or truly a circle, it has the alleged properties. It is not necessary that the figure be accurately drawn, or drawn at all, or even drawn in the imagination; if the true figure be *anywhere*, it possesses the properties. The roughest draft will answer for the proof, if we only *suppose* it to be the ideal figure.

The actual construction, therefore, by a problem to find the bisector, for instance, of an angle, or the point of bisection of a line, or a parallel through a given point to a given line, cannot be necessary to the validity of any reasoning as to theorems into which these lines and this point enter. It is sufficient to know that they exist, and on that knowledge to take lines and a point like the realities to represent them, and then to carry on the reasoning.

The *hypothesis* in all such cases may be either *directly stated* or *implied*. In the former case it is stated in the conditional form; for example, "If one side of a triangle be greater than another side, the angle opposite the greater side is greater than the angle opposite the less side." When the hypothesis is *implied*, the statement would be, "The angle opposite the greater side of a *triangle* is greater than the angle opposite the less side." The hypothesis in this latter case is that the triangle on the paper is a *real* triangle, as defined.

But it is alleged, in defence of Euclid's method, that it was necessary to make problems precede his theorems, lest any line, arc, angle, or point *supposed* in the proof of a theorem should

not exist, or be impossible to find. There is reason to believe that Euclid, in this matter, acted under some compulsion—that the sophists of his day were over-strict, and would not allow him that there *is* a bisector of a given angle or a parallel through a point to a given line until, by the necessary constructions, he had actually found them. It was a grave mistake on both sides; and the very term "postulate," under which he included three elementary problem-constructions with two supposed elementary theorems, proves that he had to *demand*, and with difficulty *obtain*, these concessions from his brother philosophers.

But the difficulty that he dreaded did not exist. The instances put forward in defence of his method are those of the impossibility of finding the trisector of an angle and two mean proportionals to two given lines. In the first place, this impossibility exists only in reference to the use of the Rule and Compass; but, by the aid of other instruments, the impossibility can be removed, as is well known. But were it impossible, by any instrument, to trisect an angle or find two mean proportionals, we know, by putting three equal angles together and by taking any four magnitudes in continued proportion, that there are *two* trisectors of every angle, and two mean proportionals to every pair of given magnitudes. And the consequence is, that we may draw a line and say, "Let this be the trisector of the angle a," and, also, "Let B and C be the required mean proportionals to A and D."

And we can, further, in this way, prove theorems as to a trisector of an angle and two mean proportionals. We can prove the conditions necessary for the trisection of an angle, and the finding of two mean proportionals, which conditions may be stated in the form of theorems.

But the case would be very different were it said, in the proving of any theorem, "Let OT be a tangent to the circle R from a point O within its circumference;" or, in the case of two non-intersecting circles, say, "Let AB be their common chord of intersection." Everybody would see that the first hypothesis is the supposition of an absurdity, and the second a contradiction. But in all other cases, where no similar difficulties existed, he would admit that it was legitimate to say, "Let the line OA be the bisector or trisector of the angle α," or, "Let Y be the parallel from the point O to the directive X," or, "Let M be the point of bisection of the line AB," without invoking the aid of any problem.

But the evil of this method of demonstration did not end here. It led to a derangement of the natural order of demonstration of Euclid's theorems. Problems taking the lead in the demonstrations, the theorems, to use a well-understood illustration, had to

"shunt" at convenient stations to allow the problems to pass on and take the lead. The inevitable consequence was, that the natural order of demonstration of the theorems was deranged, and, as Antoine Arnauld remarked of the geometricians, "disregarding the rule of method of the day, which is always to begin with things the most simple and general, in order to pass from them to those which are more complex and particular, they confuse everything, and treat pell-mell of lines and surfaces, triangles and squares, proving, by figures, the properties of simple lines, and introducing a mass of other distortions which disfigure that beautiful science."

For the reasons given above, the problems are placed at the ends of the several chapters, excepting the first, in which there is hardly material for a problem. And they are so placed, also, for another reason, that the true use of a problem is in Geometric Drawing practised on the basis of the principles taught in the theorems.

Note 3.—Areas.

A special chapter has been given to areas, with a view to assimilate the treatment of the right line and circle to that of the higher curves, conic sections, spirals, &c., and to impress on the student the distinction between the areas of closed figures and the figures themselves, which are mere outline or form. In the geometry of the higher curves, conic sections, for example, when the word "ellipse" is used it is well understood that the term applies only to the curve, not to the area bounded by it. It *has* a plane area, but that area is no part of the ellipse proper; and this holds good of the circle, which is but a particular ellipse. If Modern Geometry be right, Euclid's definition of a circle is wrong. He defines it, "A plane figure surrounded by one line called the circumference." The plane area, then, is *the* circle, and the circumference only the boundary. And this distinction as to all plane figures is frequently impressed on pupils learning Euclid; for example, when they are told that the pupil is to observe that a triangle is the space bounded by the sides, and not the outline of the sides themselves.

But it may be urged that the idea of an ellipse includes its area. If so, the ideas of the hyperbola and parabola, which are curves of the same order, cognate to the ellipse, must include their areas. But where are their areas? They have none but that which is infinite and inconceivable. Whatever view may be taken of the ellipse, it is plain that a hyperbola or parabola can be only a pure curve. The mistake consists in supposing that, because a curve encloses a space, it is no longer a curve. Then, as to other

curves, the question may be put:—Does geometry teach that the logarithmic spiral is an area?

Further, Analytic Geometry, to a knowledge of which the student of elementary geometry may arrive, confirms the views here maintained. There, the equation of a circle holds good only for the outline of a circle, and not for any point within or without it; and, for that reason, it is called the equation of *the* circle. And so for all other curves, ellipses, spirals, &c., their several equations hold good only for the curves, not for any areas. Is, then, the *elementary* geometry of the right line and circle to be in discordance with what the pupil has to learn in his after progress in Analytic Mathematics? It does not seem either right or expedient; and for that reason we have carefully distinguished between figures and their areas. In the fifth and sixth chapters frequently, for brevity-sake, the words "rectangles" and "squares" are used to denote their areas; but the preliminary observations, the context and the nature of the subject make the meaning quite evident.

Note 4.—Theor. 7, Chap. V.

This theorem might be proved from the first in this chapter, by supposing the sum of the lines in the latter to represent the sum of the squares in the former, and the difference of the lines to represent the difference of the squares, and by then applying the conclusion of the one to the other. But this would be, after all, but a covert way of demonstrating theorems algebraically; for we have no ground, geometrically, to conclude that, because a certain theorem is true about *lines*, it must be necessarily true about other orders of magnitude—squares, or cubes. For this reason, and in order to have the demonstration strictly geometrical, a distinct and simple proof has been given of each theorem.

Note 5.—Semichord of a Point.

The expression "semichord of a point," and the line that it represents, requires some notice. It has been proved (*Theor.* 5, *Chap. II.*) that only one chord through any point within a circle can be bisected at that point, and that (*Theor.* 4, *Chap. II.*) that chord is perpendicular to the diameter through the point. The chord, therefore, is *unique*, and lies symmetrically with reference to all chords through the point; and its square, as proved in this theorem, is equal to the rectangle under the segments of these

chords. It deserves, therefore, a distinctive name, such as is here given—especially as it is marked by the peculiarity that it occupies the position in reference to a point within the circle that the tangent from a point without does in reference to that point, both having the common property that their squares are equal to the rectangles under the segments, internal or external, of all chords passing through their points. The semichord is, for this reason, sometimes called the "imaginary tangent."

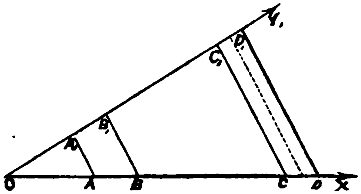
NOTE 6.—SUBMULTIPLES.

The example of Bishop Elrington, in his edition of Euclid, has been here followed in making the test of a proportion depend on the submultiples of the magnitudes involved; and for the reason that it is the simpler test, more easily comprehended by the student. No doubt the submultiple test is open to the objection raised by its application to cases involving incommensurable quantities, from which Euclid's, though very difficult to realize in the abstract, is free. But, as Elrington observes in reference to Euclid's definition, "there is not the least resemblance between it and the common notions of similitude or equality of ratios." Moreover, as it is almost certain that Euclid wrote his "Elements," not for boys, but for grown-up, hard-headed, thinking men, who could realize his definition, it is desirable to have the youth of the present age taught the doctrine of proportion by means of a definition more easily comprehended, more like the reality, and, as Elrington says, "nevertheless accurate."

It is, however, desirable that every student of geometry should, if possible, thoroughly understand Euclid's test. If the suggestion thrown out in this treatise (page 111), of a truly geometric doctrine of proportion derived from that of parallel lines, be admitted here, merely for sake of illustration, as self-evident; then the correctness of Euclid's test can be immediately made evident from it, as follows :—

Let X and Y, in the figure, be two directives, intersecting at O, and AA_1 and BB_1 two parallels meeting them. Then, assuming, as suggested, that OA is to OB as OA_1 to OB_1, we have four proportionals. The following theorem will then hold good.

If four magnitudes be proportional, and any equimultiples *whatsoever* of the first and third be taken, and also any equimultiples *whatsoever* of the second and fourth be taken, then the equimultiples of the first and third *together* exceed, or are equal to, or are less than the equimultiples of the second and fourth.



Let OC and OC_1 be any equimultiples whatsoever of the first OA and the third OA_1. Also, let OD and OD_1 be any equimultiples whatsoever of OB and OB_1; and let, also, the points A, B, C, and D be joined respectively with A_1, B_1, C_1, and D_1. Then, since OC is the same multiple of OA as OC_1 is of OA_1, the line OC is to OA as OC_1 to OA_1; and therefore CC_1 is parallel to AA_1. Also, since OD is the same multiple of OB as OD_1 is of OB_1, the line DD_1 is parallel to BB_1. But (by the assumption) AA_1 and BB_1 are parallel; and therefore AA_1, BB_1, CC_1, and DD_1 are parallel to each other. Hence, the line CC_1 being within the triangle DOD_1, the equimultiples OC and OC_1 of the first and third are *together* less than the equimultiples OD and OD_1 of the second and fourth. Also, if CC_1 coincide with DD_1 (suppose on the dotted line), the equimultiples of the first and third are *together* equal to those of the second and fourth. But if CC_1 change places with DD_1, then the equimultiples OC and OC_1 are together greater than OD and OD_1. Thus the theorem is proved.

Now, let the order of reasoning be reversed; and the converse, Euclid's definition, will be established. For, suppose OA, OB, OA_1, OB_1 not known to be proportional. If it be in any way ascertained that any equimultiples whatsoever, as previously stated, OC and OC_1 of OA and OA_1 together exceed, or are equal to, or less than OD and OD_1, any equimultiples of OB and OB_1—then CC_1 and DD_1 must be parallel to each other, and consequently AA_1 and BB_1 must be parallel, and OA, OB, OA_1, OB_1 be proportional, and the following statement hold good:—

If there be four magnitudes, and any equimultiples *whatsoever* of the first and third being taken, and any equimultiples *whatsoever* of the second and fourth being taken, if the equimultiples of the first and third *together* exceed, or are equal to, or are less than the equimultiples of the second and fourth, then the four magnitudes are proportional.

It should be noted that, since CC_1 and DD_1 are parallel to each other, and also to AA_1 and BB_1, the excesses CD and C_1D_1 of OD and OD_1 over OC and OC_1 are proportional to OA and OA_1, or to OB and OB_1. And this proportionality is but another form of the condition that the four magnitudes be proportional, for C_1 in that case could not pass D_1 without C passing D.

It is, moreover, evident that, were C_1 without D_1 while C was within D, then CC_1 and DD_1 should intersect; and consequently AA_1 and BB_1 could not be parallel to each other, and the lines OA, OB, OA_1, OB_1 could not be proportional. Further, though C and C_1 were both within DD_1, but CC_1 not parallel to DD_1, the magnitudes then also could not be proportional.

It may be well to have the above conclusions confirmed by Algebra. Let x, y, x_1, y_1 represent any four magnitudes, such as OA, OB, OA_1, OB_1, not known to be proportional; and suppose it required to determine the conditions necessary that they should be proportional. Then, m and n being any two *integer* numbers, and p and p_1 the excesses of the multiples ny and ny_1 over mx and mx_1, we have—

$$(1.)\quad mx = ny - p$$
$$(2.)\quad mx_1 = ny_1 - p_1$$

Divide (1) by my and (2) by my_1, and we have—

$$(3.)\quad \frac{x}{y} = \frac{n}{m} - \frac{p}{my}$$
$$(4.)\quad \frac{x_1}{y_1} = \frac{n}{m} - \frac{p_1}{my_1}$$

Subtract these equations; then—

$$(5.)\quad \frac{x}{y} - \frac{x_1}{y_1} = \frac{1}{m}\left(\frac{p_1}{y_1} - \frac{p}{y}\right)$$

In like manner, dividing (1) by nx and (2) by nx_1 we have—

$$(6.)\quad \frac{y}{x} - \frac{y_1}{x_1} = \frac{1}{n}\left(\frac{p}{x} - \frac{p_1}{x_1}\right)$$

If, then, the four quantities be proportional, the conditions must hold good, namely—

$$\frac{p}{y} = \frac{p_1}{y_1} \text{ and } \frac{p}{x} = \frac{p_1}{x_1}$$

which may be thrown into the form—

$$\frac{p}{p_1} = \frac{x}{x_1} = \frac{y}{y_1} \text{ or } \frac{p}{p_1} = \frac{mx}{mx_1} = \frac{ny}{ny_1}$$

The interpretation of these two equations is evident. It shows that, in order that the four magnitudes should be proportional, the excesses CD and C_1D_1 of OD and OD_1 over OC and OC_1 must be to each other in the ratio of OA to OA_1, and also in the ratio of OB to OB_1, and consequently that the four lines AA_1,

BB_1, CC_1, and DD_1 must be parallel—the same result as that geometrically obtained.

It is worth while to notice the several forms into which the tests of proportion may be thrown. Supposing OC, OD, OC_1, OD_1 to be the four proportionals. Then, OA and OA_1 are equisubmultiples of the first and third, OC and OC_1. Also, OB and OB_1 are equisubmultiples of OD and OD_1. The lines AA_1 and BB_1 are also parallel to each other. Hence Euclid's test may be converted into the following:—

If there be four magnitudes; and any equi*sub*multiples whatsoever of the first and third being taken, and any equi*sub*multiples whatsoever of the second and fourth being taken, if the equi*sub*multiples of the first and third together exceed, or are equal to, or are less than the equi*sub*multiples of the second and fourth, the four magnitudes are proportional.

Similarly, the first test of proportion (Elrington's) given in this treatise (page 113) may be converted into a test by multiples, thus:—

Four magnitudes are proportional when a multiple of the first contains the second as often as the equimultiple of the third contains the fourth.

The second test given at page 114 may similarly be converted—

Four magnitudes are proportional when any equimultiples of the first and second are also equimultiples of the third and fourth.

This is evident in itself, and is represented in the figure by making OA and OA_1 the first and second terms of the proportion, and OB and OB_1 the third and fourth, and by, moreover, supposing CC_1 and DD_1 to coincide.

NOTE 7.—INCOMMENSURABLE QUANTITIES.

The difficulty as to incommensurable quantities in reference to proportion on the submultiple system is more imaginary than real, caused by the impossibility of expressing, by means of our numeral system—or, indeed, by any numeral system—the magnitudes of certain lines which in geometry itself may be accurately represented. Nothing can be more simple and definite than the geometric relation of the diagonal of a square to its side or of the circumference of a circle to its diameter. Being given, in either case, one of the magnitudes, by a very simple construction the other is definitely found. Yet we cannot, in numbers, express accurately the magnitude of

the diagonal of a square when the magnitude of its side is given in numbers, or the magnitude of the circumference of a circle when the numerical magnitude of its diameter is given. The difficulty arises solely from the inadequacy of number to represent geometric magnitudes in all their relations.

The consequence is, that in certain cases it is impossible to express accurately in numbers the ratios of the antecedents and consequents in a proportion, and thus determine whether the ratios are equal. The difficulty is purely arithmetical.

Referring to the same Figure as before (*page* 163), let OC, OD, OC_1, and OD_1 be four proportionals; and suppose that OC and OD are incommensurable, and OC_1 and OD_1 consequently similarly incommensurable. Take, then, any submultiple OA of OC and the equisubmultiple OA_1 of OC_1; and join AA_1. Let, also, OA and OA_1 be taken on OD and OD_1 respectively as often as they will go. Then, since the pairs of magnitudes are incommensurable, there must be remainders. Let the distances xD, x_1D_1 of the extremities (which we may denote by the letters x and x_1) of the dotted line from D and D_1 be these remainders; and we have (*First Test of Proportion*) Ox to OC as Ox_1 to OC_1, and the dotted line is parallel to both AA_1 and CC_1. Take, now, any equisubmultiples, however small, of OA and OA_1; they will evidently also go into Ox and Ox_1 without remainder; and if they go any number of times into xD and x_1D_1, then the dotted line will be moved closer to DD_1. If they do not go into xD and x_1D_1, but are greater than them, then smaller submultiples must be found by repeating continually the division of the preceding submultiples until the dotted line moves on gradually by frequent divisions towards DD_1, and eventually coincides with it. Then the infinitesimal submultiple of the first OC is contained in the second OD as often as the infinitesimal equisubmultiple of the third OC_1 is contained in the fourth OD_1; and the test holds good.

There is really no difficulty in this application of submultiples which an ordinary student cannot master. An infinitesimal cannot be pictured; but it is conceivable. And it can, moreover, be understood that infinitesimals have different magnitudes; for, in the illustration here given, it is evident that the submultiples in their successive divisions maintain the ratio to each other of OC to OC_1.

GEOMETRIC EDUCATION.

The Platonicians of old wrote over their school an inscription forbidding any to enter who were not geometricians. This, to use a modern term, was their condition of matriculation. It is related of Xenocrates, also, that he refused a young man, who was ignorant of geometry, astronomy, and music, admission to his classes, on the ground that he had not the "handles of philosophy." There must have been some deep conviction in the minds of such great thinkers as to the educational value of these sciences, lost sight of in modern times, which impelled them to seek such qualifications in their pupils. Whatever the conviction was, it holds good of general education, as well as of philosophic; for every educated gentleman is, in his degree, a philosopher—the learned few of antiquity have become the educated many of these days. The love of knowledge is common to both. True philosophy is education; and all education should be truly philosophic. But what did these Greeks mean when they declared the geometric sciences "λαβὰς εἶναι φιλοσοφίας?" Assuredly, that they open the eyes of the mind and enable it to see beauty and order, where otherwise all would be confusion. Astronomy, as the ancients studied it, was but little more than the geometry of the sphere, applied to a few material bodies moving in space looked at from this earth as a central position. Music is harmony, the order of sounds, with which the soul sympathizes. And to these may be added, as another philosophic "handle," arithmetic, the science of number, which, taken in both its abstract and concrete forms, together with geometry, embraces the whole range of the Pure Mathematics. Indeed, in these days, the importance of arithmetic is not questioned. From the infant-school to the pinnacle of a fellowship, it rules as an essential of education with undisputed sway. But this high estimate of its value is due much more to its being a handle of commerce than of philosophy. Commercial arithmetic, however, has its educational aspect; and Mammon, in his own way, although indirectly, has helped to make the philosopher. It is the science of number, and the earliest educator of the reason. It lifts man from the condition of the savage, who can describe a multitude only by pointing to the hairs of his head. It converts

the chaos of units into an orderly host. There ought to be a similar power in Geometry to bring beauty out of disorder. It is the science of Form; and, inasmuch as form comprehends most that is beautiful and attractive to the eye—most of what is charming in imagination, which might well be described as departed vision recalled to life and etherealized—its influence ought to be greater than that of arithmetic—greater, were it but rightly apprehended and taught in, at least, all mental training, the aim of which is refinement.

And geometry, in its first stages at least, ought not to be hard to learn. There is the eye to help—an advantage which arithmetic, easily learned enough, does not, to the same extent, possess. The student has his handle as well as the philosopher; and he who would teach him rightly must take him by the right handle. Truly, there are fewer dunces in this science than is commonly imagined. There are some persons who are incapable of learning any abstract science; there are others that can push through every one of them with gigantic strides; but the vast majority—average human nature—are capable of perceiving the relations of form as they are those of number, or of enjoying the harmony of sounds. It has been often said that geometry is valuable in training the mind to reason accurately, and no doubt it is; but we must remember that its proper function is not to teach logic, but the relations of form and figure. And, even as to the culture of the reasoning faculty which it affords, he who pursues it on account of form and figure will derive more advantage from it as a reasoning exercise than he could through any indirect motive.

But, in order to enter on the study of geometry with a prospect of success, the class of pupils who constitute the οἱ πολλοί of our schools must begin with their eyes, not with intricate reasoning processes. There must, from the commencement, be reasoning; but it should be of the simplest kind, concerned about the relations of lines, angles, &c., which can be made visible, as it were. Education in geometry naturally divides itself into two stages—the first, where the reasonings are simple, and sense plays an important part; the second, where geometric facts have accumulated, imagination has displaced sense, and the reasoning has become more varied, abstract, and complex. The latter is True Geometry, the former, the scaffolding, which may be cast away when once the edifice is erected.

But all *elementary* treatises on geometry are, more or less, scaffolding. There is, first, the foundation laid in the definitions, as to which great care should be taken that they be properly explained and well illustrated by a reference to popular examples, in order that the beginner may not, at the outset, be

thrown into confusion by misconceptions. This should be done on the gallery, with the aid of a black-board, or of diagrams, and not by the old-fashioned method of giving tasks to be learned by the pupil previous to explanation. The basis being thus laid, a few apparatus will wonderfully help the young geometrician as he approaches the theorems. It seems almost puerile to speak of a few rods, models, or diagrams, as of value in teaching this *abstract* science in schools; but the truth is, that those who would teach children must become as children, and descend from the high dignity of abstraction, even though geometry be offended. The appliances may be vulgar, but the end will justify the means.

Taking this view of his art, the teacher will do well to have models of triangles and polygons in wood to exemplify and illustrate all demonstrations which are effected by superposition or juxtaposition. The principle of superposition is invaluable, for it is an appeal to the eye as well as to the reason; and, when applied in this way by a skilful teacher, it presents no difficulty to most pupils. Juxtaposition, used in the third chapter of this work, is even more valuable. It prevents, at least, that confusion of the pupil's mind which, in superposition, is frequently caused by one of the two things made to coincide being lost for the moment to the eye. In juxtaposition the appeal is to the isosceles triangles, which are created by it, as exemplified by the Theorems 8, 9, and 10 of the third chapter. The isosceles triangle is useful for these visible demonstrations. To turn it to good account the teacher should have a model circle cut in hard wood, divided and hinged in the line of a diameter, to illustrate the theorems of superposition, the first and second of the second chapter, that equal arcs of the same circle have equal chords and subtend equal angles at the centre; and that a perpendicular from the centre on a chord bisects the chord and the arc, and that the radii to the extremities of a chord make equal angles with it. There should be also, a parallel ruler on a large scale to illustrate the properties of parallel lines and parallelograms; and a parallelogram in hard wood, divided along a diagonal and hinged so that the two triangles into which it is divided may be placed over each other, or unhinged, to exhibit, separately, the two triangles of which it is composed. By these means, and a few transversals, theorems on parallel lines may be illustrated. Models, also, to illustrate Theorems 8, 9 and 10 of the third chapter, are desirable, hinged along the line of juxtaposition of the triangles. The equality of the areas of figures, as set forth in the theorems of the fourth chapter, may also be illustrated by such models—also the equality of the areas in the several cases of triangles, and those of segments and sectors of circles.

The extent to which this teaching by models may be carried is

an important question. The danger is that it may be made a substitute for the strict reasoning process which is essential to geometry. Their proper use is to open the eyes, and, therefore, for that reason, they should be confined to the first chapters of a treatise on the subject. After that stage, the imagination is brought more into play, the reasoning becomes more and more abstract, and the hypothetic character of the conclusions are better understood. The judicious teacher, if his method be good, will feel his way in any case of real difficulty. The principle is one and the same for all, namely, in the first ascents of the geometric ladder, to keep the eyes open, and that once done, not to offend the reason.

But some may think that no material aid should be given, even in the earliest stages; that, were it given, rigour and exactitude of demonstration would be at an end. No; it must be as rigorous at the first step as at the last—the reasoning run in one thread all through. The apparatus suggested are not intended to be substitutes for reasoning, but helps to enable the pupil to reason better. Exactly as a telescope is not a substitute for a knowledge of the stars, but a means whereby we may know them better. To the pupil of medium ability, models and diagrams are means to an end—spectacles to aid a weak vision, not required by the clear-sighted, but valuable to *him*. But a theorem will now and then present itself where this extraneous aid will not avail, the reasoning of which may prove too intricate. Let us take, for instance, the fourth case of equal triangles (*Theor.* 11, *Chap. III.*), the reasoning in which is rather intricate. What is to be done? Let the pupil pass over the difficult theorem *for the time being*; and, assuming it to be true, proceed to the next. This is what a judicious teacher does in other subjects; not meaning thereby that the pupil should *never* afterwards learn the proof, but that he *should*; that is, returning to it when he is stronger, should *then* learn it. And this directs attention to a remarkable fact connected with the mathematical education of this country, which is not commonly noticed. It is curious that one of the branches of mathematics, geometry, is taught in schools on a principle the very opposite of that which is adopted in teaching arithmetic and algebra. It is not considered necessary, in teaching these latter subjects, to rack the pupil's brain with the reason of every rule he learns. The tendency is to the opposite extreme—to require *no* reasons at all, and reduce education in these sciences to learning certain rules by heart, and mechanically working them out in practice. And another singular circumstance is, that very often the teachers who insist that not a link of the chain of abstract reasoning be left out by the infant who tries to master one of Euclid's propositions, is often found condemning, as utterly useless, the requiring of the

same pupil to know the reason of the rules he works in algebra. He allows him, in darkness as to the reason of the mysterious operation he is performing, to solve a quadratic equation, find the greatest common measure, or the value of a circulating decimal; but he must not venture to believe the palpable truth that the base angles of an isosceles triangle are equal until he has, with the nicest accuracy, strung together the legs and angles of at least five triangles. Now, these are two extremes of error. They prove that education, taken either as a science or an art, is yet in its infancy. Such education is unphilosophic; and we are yet, as regards that important instrument of progress, behind Xenocrates and the Platonicians.

There is a *viâ media* which to follow is wisdom in this, as in all other branches of education, namely, to train the infant reason, but not to strain it—to teach truths to some with the reasons, to others without them, reserving the proofs in the latter case for a more convenient season—beginning the geometric training of the mind in school early, carrying it on slowly but surely, and so conducting it with judgment that, by the time the youth is ready to matriculate, he may know such a book as Euclid with *all* its *reasons*, and approach the portals of his university without fear of exclusion, even though the inscription written above be ὀυδεὶς ἀγεωμέτρητος ἐισίτω.

The teaching of problems calls for a few remarks; and then we are done. It is the practical part of the science, and stands in the same relation to the theorems that sums in arithmetic do to the principles on which the rules by which they are worked are based. As geometry is taught in Euclid's pages, the true nature and importance of the problem is lost sight of, it being there made no more than a step to, or a link in, the demonstration of the theorem. Teachers have often noticed that pupils at first take more readily to problems than to theorems. The operations are mechanical, the results tangible, and the reasoning is confirmed by the eye. And, if it be true that sight is an important ally in teaching geometry, the less clever portion of all school classes should be largely exercised at problems. The effort to make the constructions helps to give geometric ideas. A perception of the relations of the parts of the constructed figures grows on the mind; and thus difficulties in theorem-demonstration, which at first seemed insurmountable, often of themselves gradually disappear. For example, if a pupil has been exercised in constructing the cases of triangles, after having lightly gone over the theorems relating to them, the insight he will get into the details of the figures will, when he turns back on the theorems, enable him to understand superposition, juxtaposition, or any other kind of proof of these cases, much more easily than he could have done before. This is nothing but common sense; the

result of experience in every other branch of education—the languages and the natural sciences. It is also the course taken in other departments of mathematics; and why should it not be the right course in geometry? Each theorem should, therefore, as soon as possible, be thrown into this practical form; and the light obtained by the operation be allowed to reflect back on it. Moreover, all these constructions should be made with accuracy, for the pupil should be allowed the satisfaction of seeing that the result at which he has arrived is *like* what it professes to be.

Against all this problem exercise, it may be urged that accuracy in doing sums in arithmetic is of practical utility, whereas no such reason can be given for this problem exercise. There is, no doubt, this difference in the two cases; but there is also a resemblance. And on this resemblance, not that difference, our remarks are based, namely, that practice in every science throws light on principles. The most the objection proves is, that there ought to be *more* exercise in arithmetic than in geometry, by no means that there should be *none* in the latter.

But there is a case exactly in point. It is acknowledged that making Latin verses in school is not enforced with a view to making Englishmen Latin poets. The amount of school-boy energy consumed annually on Latin verses is enormous; but the justifying reason given for the expenditure is, that practice throws light on principles—that the pupil is helped by it to know, to speak, and to write his Latin better. This is all that is here claimed for exercises in geometric problems. But there is this difference in the two cases, namely, that one is a beaten track that it is hard to get people out of, whereas the other is a new one that it is equally hard to drive them into.